# Buyer-Optimal Extensionproof Information*

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**Abstract**

We study buyer-optimal information structures under monopoly pricing. The information structure determines how well the buyer learns his valuation and affects, via the induced distribution of posterior valuations, the price charged by the seller. Motivated by the regulation of product information, we assume that the seller can disclose more if the learning is imperfect. Extensionproof information structures prevent such disclosure, which is a constraint in the design problem. Our main result identifies a two-parameter class of information structures that implements every implementable buyer payoff. An upper bound on the buyer payoff where the social surplus is maximized and the seller obtains just her perfect-information payoff is attainable with some, but not all priors. When this bound is not attainable, optimal information structures can result in a higher payoff for the seller and in an inefficient allocation.

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# 1 Introduction

Before making a purchase decision, consumers typically try to assess how well the product under consideration matches their preferences, using various sources of information. Examples include technical specifications or a list of ingredients published by the seller, advertising, reviews, product samples, and testing the product during a trial period. Whereas sellers often have considerable control over such information, its disclosure is regulated in many countries, with the aim of promoting consumer welfare. The European Union, for example, has passed regulation ranging from food information over insurance mediation to the content of financial security prospectuses. It has also introduced a mandatory period of 14 days during which consumers can withdraw from a sales contract concluded via the Internet.[1] Effectively, this period amounts to a trial period during which consumers can learn better to what extent the product fits their preferences.

Sellers are usually free to provide more information than the regulator requires. A trial period, for instance, can be extended beyond the obligatory number of days.[2] When setting minimum disclosure requirements, the regulator must therefore take into account how the requirements affect sellers' incentives to disclose more. More information is not necessarily advantageous for buyers: it allows better purchasing decisions, but if the information creates more dispersion in the buyers' willingness to pay, sellers may raise prices. Hence, what are buyer-optimal minimum disclosure requirements when the seller can disclose more? This is the question we address in this paper.

We take an information-design approach and study buyer-optimal information structures under monopoly pricing. In our model, the seller has a single object for sale, which she values at zero, and she faces one potential buyer. The seller and the buyer have a common prior about the buyer's valuation for the object, which is unknown to both of

[1]See, respectively, Regulation (EU) No 1169/2011, Directive 2002/92/EC, Regulation (EU) 2017/1129, and Directive 2011/83/EU.

[2]For example, in the European Union, the Apple online store accepts returns within the obligatory 14 days, whereas Amazon extended this period to 30 days, Zalando, an online fashion retailer, to 100 days, and IKEA to a full year.

them. An information structure consists of a set of signals and probability distributions over signals conditional on the buyer's valuation. At the outset, the buyer (or a regulator) chooses an information structure. Thereafter, the seller sets a price and decides about releasing additional information. Specifically, she can *extend* the information structure by adding a signal component. At the end, the buyer privately observes the signal of the (possibly extended) information structure, updates to a posterior valuation, and decides whether or not to buy. Since any additional signal component can be incorporated at the outset, we restrict attention to information structures under which the seller has no incentive to disclose more. We call such information structures *extensionproof*. Accordingly, we study the buyer's (or regulator's) problem subject to the constraint that the information structure is extensionproof.

The core assumption of our model is that the buyer cannot commit to ignore any additional information released by the seller: such commitment power would eliminate the extensionproofness constraint. Once the seller has fixed the price, the buyer can only benefit from using any additional information when making his purchase decision. The seller may thus question the credibility of any claim to ignore information, making it natural to assume that the buyer lacks commitment. A further motivation for this assumption is that the regulation of product information is prevalent for consumer goods, where the buyer's identity is largely anonymous and the terms of trade are standardized. An anonymous buyer cannot make the seller aware of his intention to ignore information, and even if he could, this would not change the good's standard price.

In our model, any information that the buyer obtains is private—the seller discloses without observing as in the seminal contributions by Lewis and Sappington (1994) and Eső and Szentes (2007). For illustration, consider a trial period that enables the buyer to partially learn his willingness to pay by experimenting with the product. Choosing the length of the period gives the seller control over the accuracy of the buyer's learning, but what the buyer learns is his private information. Similarly, based on the features highlighted to him by the insurer, a potential insuree privately determines to what extent the insurance contract is appropriate for his personal situation. More generally, a consumer's assessment based on product information is typically governed by an idiosyncratic learning process that is not perfectly predictable by the seller.

Our model makes no restrictions on the information design. Specifically, every distribution of posterior beliefs that is consistent with Bayesian updating can be attained. For example, the information structure may allow more precise learning about extreme rather than intermediate valuations, or the seller's extension may be more informative depending on what the buyer knows already. Indeed, regulators and sellers often have considerable control over the buyers' learning. For instance, when viewing the good as a bundle of product attributes (such as the camera, battery life, and screen quality of a smartphone) and assuming lexicographic preferences, then disclosing only the key attributes will make those who discover a strong liking or dislike more confident as to their willingness to pay than others. Moreover, the importance of information about one attribute can depend on what else is disclosed (e.g., information about the cases available for a smartphone upon inspecting its look and feel). Digital products may permit particular flexibility. For example, the buyers' learning about a software application can be fine-tuned through a careful choice of the functionality of the trial version they are given access to.

The following insights on buyer-optimal regulation of product information emerge from our analysis. First, learning the valuation perfectly is typically not optimal for the buyer even though imperfect information must be safeguarded against extensions. Second, the extensionproofness constraint always binds, that is, the seller's incentives to disclose more must always be taken into account. Third, the most effective deterrence of harmful extensions requires the disclosure of anything that helps the buyer assess the relevance of any information that remains concealed, which results in the buyer deeming more or less extreme valuations likely depending on the inferred relevance. Fourth, there can be different allocative consequences: In some markets, the buyer-optimal regulation also maximizes the social surplus and only grants the lowest implementable payoff to the seller, which is what she could secure by extending to perfect information. In other markets, however, it sacrifices some social surplus and grants a higher payoff to the seller, who thus strictly prefers being regulated over perfect information. Moreover, all these insights extend to situations where the seller can screen the buyer sequentially.

Our main result, on which these insights largely build, identifies a two-parameter class of information structures with the property that for every buyer payoff that can be implemented by *some* information structure, there exists an information structure in this class that implements this payoff. The information structures are characterized as follows. The two parameters determine an interval of valuations. All valuations outside this interval are disclosed perfectly. All valuations inside it are pooled, pairwisely and such that the posterior valuation is always the same. In particular, the pooling proceeds in a deterministic, negative assortative fashion: high valuations are pooled with low ones according to a decreasing matching function.

In the derivation of this result, we exploit a connection to matching, or optimal transport. We consider the problem of inducing a given buyer payoff while minimizing the seller's gain from disclosing more. We confine this problem to information structures that pool only the valuations inside some interval, pairwisely and such that the posterior valuation is always the same. Here, the pooling might still be stochastic. The key step is to establish an equivalence between such information structures and a certain class of all bivariate distributions with given marginals. Working with the bivariate distributions, we get an optimal-transport problem. This problem has a supermodular objective function, which implies that pooling in a deterministic, negative assortative fashion is optimal.

The main result narrows the search for buyer-optimal information structures down to the two parameters of the negative assortative information structures. A natural upper bound for the buyer payoff is given by trade with probability one, maximizing the social surplus, and the seller getting just her perfect-information payoff, which she can always secure by disclosing perfect information. Through the restriction to negative assortative information structures, we obtain a characterization of the priors with which this upper bound is attainable. When the bound is not attainable, optimal information structures can result in the seller getting a strictly higher payoff than under perfect information and, at the same time, in a probability of trade strictly less than one and thus an inefficient allocation. But negative assortative information structures are constrained efficient: for any given buyer payoff, they induce the highest possible corresponding seller payoff.

Our analysis contributes to the literature on information design (e.g., Kamenica and Gentzkow, 2011; Bergemann, Brooks, and Morris, 2015; Li and Shi, 2017). The most closely related paper is the one by Roesler and Szentes (2017), who also study buyer-optimal information structures under monopoly pricing but without disclosure by the seller. Their results provide a benchmark for evaluating the relevance of our extensionproofness constraint. The constraint always binds: unconstrained optimal information structures yield the seller even less than her perfect-information payoff. Like us, Roesler and Szentes identify a class of information structures that implements every implementable buyer payoff. We show that their class need not contain an optimal information structure for our setting. In both settings, optimal information structures typically do not remove the buyer's uncertainty completely (see also Kessler, 1998).

Several recent papers also study information structures that pool types in a negative assortative fashion. Von Wangenheim (2017) shows that the same class of information structures as here implements every implementable combination of buyer and seller payoff in sequential screening.[3] The key difference is that the buyer eventually learns his valuation perfectly, whereas in our paper the seller endogenously decides how much information to add. Nikandrova and Pancs (2017) consider sequential two-bidder auctions with information acquisition. When recommending information acquisition to the second bidder, the auctioneer optimally pools high and low bids of the first bidder to mitigate incentive constraints. Goldstein and Leitner (2018) and Garcia and Tsur (2018) show that the optimal disclosure policy of an informed regulator may feature negative assortative pooling of banks in financial markets and of risk types in insurance markets, respectively. Studying a dynamic model of cheap talk, Golosov, Skreta, Tsyvinski, and Wilson (2014) construct equilibria that involve negative assortative pooling and improve communication compared to the static model. The idea that information structures need not pool more than two states into a given message is also used by Kolotilin (2018), who furthermore describes a pooling of high states with low ones.

Li and Norman (2018) study a general persuasion game where, as in our model, several players can disclose information sequentially (see Gentzkow and Kamenica, 2017,

[3]We thank Jonas von Wangenheim for pointing us to this class.

for simultaneous disclosure). Like here, attention can be restricted to equilibria in which subsequent players have no incentive to add information. Similarly, Perez-Richet and Skreta (2018) consider a model of test falsification and show that optimal tests can be found among falsification-proof ones.

Concerning persuasion of a privately informed receiver, Kolotilin, Li, Mylovanov, and Zapechelnyuk (2017) establish a payoff equivalence between experiments in which the disclosure is independent of the receiver's type and mechanisms in which the disclosure depends on a report by the receiver (see also Guo and Shmaya, 2019). For our main result, we assume that the seller's extension can condition directly on the signal from the original information structure. We show that if the buyer observes the signal before the seller decides about providing an extension, mechanisms in which the extension and the terms of trade depend on a report are suboptimal for the seller.

While our focus is on buyer-optimal information structures, another strand of literature on information design studies *seller*-optimal information structures for various selling environments (see, e.g., Lewis and Sappington, 1994; Bergemann and Pesendorfer, 2007; Eső and Szentes, 2007; Board and Lu, 2018). The buyer in our model has no private information at the outset, and to maximize the social surplus, he should always get the object. Thus, the seller-optimal information structure would simply provide no information. A large and influential literature investigates the incentives of sellers to voluntarily disclose information that is objective (i.e., everybody can assess its relevance) and certifiable (i.e., the seller can prove the true state). According to the "unraveling" argument (Grossman and Hart, 1980; Milgrom, 1981), sellers automatically have an incentive to disclose such information. In our model, the argument does not apply: the relevance of the information to the buyer depends on the buyer's individual preferences, which the seller does not know (see also Koessler and Renault, 2012).

The rest of the paper is organized as follows. The next section presents the model. Section 3 shows that becoming perfectly informed is typically not optimal for the buyer. Section 4 illustrates our results for a uniform prior. In Section 5, we establish the main result on negative assortative information structures. Section 6 studies buyer-optimal information structures. In Section 7, we discuss general mechanisms, a weaker extensionproofness constraint, how the seller's ability do add information changes the design problem, and what happens if seller and buyer switch roles. Section 8 concludes. Most proofs are in the Appendix.

# 2 Model

**Payoffs and prior information.** A seller has a single object to sell to a buyer. The buyer's valuation for the object is initially unknown to both parties. Both believe that it is drawn from the cumulative distribution function (CDF) $F$ over $[0,1]$, which admits the strictly positive probability density function (PDF) $f$. The seller offers the object at a take-it-or-leave-it price $p$. If the buyer accepts the offer and has valuation $v$, then his payoff is $v-p$ and the seller's payoff is $p$. If the buyer rejects, payoffs are both zero.

**Information structures.** Before the buyer decides about the purchase, he receives information about his valuation. Specifically, he observes a signal from some information structure. An *information structure* is a combination $(S,(G_v))$ of a signal set $S$ and CDFs $G_v$ on $S$ such that if the buyer has valuation $v$, then a signal $s \in S$ is drawn from $G_v$ and privately observed by the buyer. A *perfect* information structure, for example, has CDFs $G_v$ whose supports are disjoint across $v$, so that it reveals the valuation fully. The signal set $S$ of an information structure is a subspace of some Euclidean space. Let $\bar{G}$ denote the unconditional CDF on $S$, that is,

$$\bar{G}(s) := \int_0^1 \int_{\{e \in S : e \leq s\}} dG_v(e) dF(v).$$

**Actions and timing.** There are three stages. First, the buyer (or a regulator) chooses an information structure $(S^a,(G_v^a))$. In the second stage, the seller observes $(S^a,(G_v^a))$ and sets a price $p$. Moreover, she decides about releasing additional information. Specifically, she can *extend* $(S^a,(G_v^a))$ to any information structure $(S,(G_v))$ with $S = S^a \times S^b$ for some $S^b$ and $\int_{S^b} dG_v(\cdot, s^b) = G_v^a$. In the third stage, the buyer observes the (possibly extended) information structure and the signal, updates his belief about his valuation, and decides whether or not to buy the object.

**Posterior beliefs and posterior valuations.** Upon observing signal $s \in S$ from information structure $(S, (G_v))$, the buyer updates his belief to a posterior distribution function $F_s$ over valuations $v \in [0,1]$. Formally, the posteriors are characterized by the condition that for all $V \in \mathcal{B}([0,1])$ and all $M \in \mathcal{B}(S)$,

$$\int_M \int_V dF_s(v) d\bar{G}(s) = \int_V \int_M dG_v(s) dF(v), \tag{1}$$

where $\mathcal{B}(\cdot)$ denotes the respective Borel $\sigma$-algebra.[4] Hence, the posterior valuation upon observing $s$ is $E[v|s] = \int_0^1 v dF_s(v)$, and so the information structure induces the CDF of posterior valuations

$$H(w) := \int_{\{s \in S : E[v|s] \leq w\}} d\bar{G}(s).$$

Note that under a perfect information structure, $H$ coincides with the prior $F$.

We assume that the buyer purchases the object if and only if $E[v|s] \geq p$. Thus, given price $p$ and a CDF of posterior valuations $H$, the (ex-ante) probability of trade is $\int_p^1 dH(w)$.[5] An information structure *induces* price $p$, buyer payoff $U$, and seller payoff $\Pi$ if $p \in \operatorname{argmax}_q q \int_q^1 dH(w)$, $U = \int_p^1 (w-p) dH(w)$, and $\Pi = p \int_p^1 dH(w)$.[6] In words, this means that without additional disclosure, the seller is willing to set price $p$ and this price results in buyer payoff $U$ and seller payoff $\Pi$. When the seller has no incentive to disclose more, we occasionally use the term *implement* instead of 'induce'.

Our aim is to study the information structures that maximize the buyer payoff when the seller can disclose more. Let $(S^a, (G_v^a))$ be any information structure, and suppose it is optimal for the seller to extend $(S^a, (G_v^a))$ to $(S, (G_v))$. Then, by the optimality of the seller's extension, $(S, (G_v))$ does not induce further disclosure. Accordingly, we confine the analysis to information structures under which the seller has no incentive to disclose more (and we usually omit the superscripts $a, b$). We call such information structures *extensionproof*.

[4]Thus, the posteriors $F_s$ are the CDFs corresponding to a regular conditional distribution, which exists and is unique almost everywhere (see, e.g., Dudley, 2002, Thm. 10.2.2).

[5]Integrals of the form $\int_a^b$ stand for $\int_{[a,b]}$. This distinction matters because distributions such as $H$ can have atoms. Here, for example, we account for the case that $w = p$ with positive probability.

[6]Where no confusion results, we write "payoff" instead of "expected payoff", and similarly for surplus.

## 3 Suboptimality of Perfect Information

First, we show that perfectly learning his valuation is typically not optimal for the buyer. Under a perfect information structure, the posterior valuation is distributed according to the prior $F$, and so the seller payoff at price $p$ is $[1 - F(p)]p$. Denote the lowest optimal price for the seller by

$$p^* := \min \operatorname*{argmax}_p [1 - F(p)]p$$

and the corresponding seller payoff by

$$\Pi^* := [1 - F(p^*)]p^*.$$

Thus, a perfect information structure induces at most the buyer payoff

$$\int_{p^*}^{1} (v - p^*) dF(v).$$

**Proposition 1.** *Suppose the PDF $f$ is continuous. Then, there exists an extensionproof information structure that induces a buyer payoff strictly greater than $\int_{p^*}^{1}(v - p^*)dF(v)$.*

To make the buyer better off than under perfect information, the information structure must implement a price $p < p^*$. This requires that the probability of trade at price $p$ is greater than under perfect information,

$$\int_p^1 dH(w) > \int_p^1 dF(v), \tag{2}$$

for otherwise the seller would gain by extending to perfect information and charging $p^*$. The only way to achieve (2) is by pooling (a nonzero mass of) valuations $v < p$ with valuations $v > p$ into signals that result in posterior valuations of at least $p$: because of the valuations $v < p$, this increases the probability of trade at $p$. Indeed, the proof of Proposition 1 shows that pooling all valuations within some interval $[\underline{v}, p^*]$ into the same signal, which results in posterior valuation $E[v|v \in [\underline{v}, p^*]] < p^*$, and perfectly disclosing all other valuations suffices to improve on perfect information under a continuous PDF.

# 4 Example: The Uniform Case

To illustrate our main results, we construct here a buyer-optimal information structure for the special case where the prior is the uniform distribution (i.e., $F(v) = v$).

Because the seller can always extend to perfect information, she must get under any extensionproof information structure at least her perfect-information payoff $\Pi^* = \max_p (1-p)p = 1/4$. The maximum social surplus is $E[v] = 1/2$, which materializes if trade happens with probability one. Consequently, the buyer payoff, which is the difference between the social surplus and the seller payoff, can be at most 1/4.

We will show that the following information structure attains this upper bound on the buyer payoff: If $v > 1/2$, display $s = v$ with probability one. Thus, the buyer learns his valuation perfectly. If $v \leq 1/2$, display $s = |v - 1/4|$ with probability one. Thus, for valuations $v \leq 1/2$ the buyer only learns the distance between his valuation and 1/4, which leads to posterior valuation 1/4. The distribution of posterior valuations is then

$$H(w) = \begin{cases} 0 & \text{if } w \in [0, \frac{1}{4}), \\ \frac{1}{2} & \text{if } w \in [\frac{1}{4}, \frac{1}{2}], \\ w & \text{if } w \in (\frac{1}{2}, 1]. \end{cases} \tag{3}$$

It is straightforward to verify that this information structure induces price 1/4, that is, $1/4 \in \operatorname{argmax}_p p \int_p^1 dH(w)$. Moreover, as trade happens at this price with probability one, the induced seller and buyer payoffs are both equal to 1/4.

We now demonstrate that the above information structure is extensionproof—the seller cannot gain by extending it. To this end, we show that there is no combination of an extension and a price $q$ that yields a seller payoff strictly greater than 1/4. Under every extension, prices below 1/4 or above 1/2 are strictly dominated by price 1/2, which just yields seller payoff 1/4. So take any price $q \in (1/4, 1/2)$ and suppose the seller chooses an extension that maximizes the probability of trade (and hence her payoff) at $q$. First note that for some valuations $v$, the signal $s$ is already sufficiently informative such that no extension can change the buyer's decision: he always buys if $v \geq 1/2$ and he never buys if $v \in (1/2 - q, q)$ (i.e., if $s < q - 1/4$). To maximize the probability of trade for the remaining valuations $v$, the seller can extend the information structure as follows: If $v \in [q, 1/2]$, display a signal $BUY$ with probability one.[7] If $v \in [0, 1/2 - q]$, display $BUY$ with probability

$$x(v) := \frac{\frac{1}{2} - v - q}{q - v}.$$

Then, upon observing $s$ and $BUY$ the buyer assigns the probabilities

$$\frac{x(\frac{1}{4} - s)}{x(\frac{1}{4} - s) + 1} = \frac{s + \frac{1}{4} - q}{2s} \quad \text{and} \quad \frac{1}{x(\frac{1}{4} - s) + 1} = \frac{s - \frac{1}{4} + q}{2s}$$

to $v = 1/4 - s$ and $v = 1/4 + s$, respectively. Hence,

$$E[v|s, BUY] = \frac{(s + \frac{1}{4} - q)(\frac{1}{4} - s) + (s - \frac{1}{4} + q)(\frac{1}{4} + s)}{2s} = q,$$

that is, the buyer's posterior valuation upon observing $s$ and $BUY$ is exactly $q$. Consequently, for any $s$, the extension persuades the buyer to buy with probability one if $v \geq q$ and with the highest possible probability (i.e., $x(v)$ or 0) if $v < q$. The seller payoff with this extension is

$$\left(1 - q + \int_0^{\frac{1}{2}-q} x(v)dv\right) q < \left(1 - q + \int_0^{\frac{1}{2}-q} \frac{\frac{1}{2} - q}{q} dv\right) q = \frac{1}{4}.$$

Hence, the information structure is extensionproof.

Note that there are many information structures that also induce the CDF of posterior valuations (3) but are not extensionproof. For example, suppose all $v > 1/2$ are disclosed perfectly and all $v \leq 1/2$ are pooled into the same signal. In that case, the seller could add the information of whether or not the valuation exceeds $1/4$. Thus the buyer's posterior valuation is $3/8$ if $v \in [1/4, 1/2]$ and the price $3/8$ yields seller payoff $(1 - 1/4) \cdot 3/8 > 1/4$. Hence, for extensionproofness the distribution of posterior *beliefs* matters, not just the distribution of posterior *valuations*.

# 5 Negative Assortative Information Structures

We now return to the general case, where the prior $F$ is arbitrary, and show that the search for buyer-optimal information structures can be restricted to a two-parameter

[7] For convenience, we occasionally use terms such as "$BUY$" for particular signals.

class of information structures, which we call "negative assortative". Every implementable combination of price and buyer payoff remains implementable when restricting to this class, along with the highest possible corresponding seller payoff. The optimal information structure for the uniform case in the preceding section belongs to this class.

To make the logic of our derivation clear, we break it down into three steps. The first step (Lemma 1) shows that every implementable combination of buyer and seller payoff as well as price can be implemented by an information structure that pools not more than two valuations into any signal and under which the posterior valuation either equals the price or the true valuation. Accordingly, we restrict attention to such information structures thereafter. The second step (Lemma 2) determines optimal extensions for the seller. The final step (Lemmas 3 and 4) shows that the problem of minimizing the seller's gain from an extension while inducing a given combination of buyer payoff and price is an optimal transport problem, which has a well-known solution.

We say that an information structure $(S, (G_v))$ is *p-pairwise* if for almost all signals $s$ there exist valuations $v_L, v_H \in [0,1]$ such that the posterior belief $F_s$ has support $\{v_L, v_H\}$ and

$$\text{either:} \quad v_L = v_H \tag{4}$$

$$\text{or:} \quad v_L < p < v_H \text{ and } E[v|s] = F_s(v_L)v_L + [1 - F_s(v_L)]v_H = p. \tag{5}$$

Thus, under a $p$-pairwise information structure the buyer deems at most two valuations possible upon observing the signal, and whenever he deems two valuations possible, his posterior valuation is exactly $p$. The buyer-optimal information structure presented in Section 4 is $p$-pairwise (with $p = 1/4$).[8]

**Lemma 1.** *For every extensionproof information structure that induces price $p$, there exists an extensionproof $p$-pairwise information structure that induces the same price, the same buyer payoff, and the same seller payoff.*

Invoking this lemma, we can restrict attention to $p$-pairwise information structures. The basic intuition is as follows. The price and the payoffs depend only on the CDF of

[8]Moreover, its signal is a deterministic function of the valuation, which need not hold for $p$-pairwise information structures in general.

posterior valuations. To deter extensions by the seller, the CDF of posterior valuations should be implemented by an information structure that is already as informative as possible. Every information structure that pools more than two valuations into the same signal can be made more informative without changing posterior valuations. For example, suppose three valuations $v' < v'' < v'''$ are pooled into the same signal $s$, where $E[v|s] \in (v'', v''')$. Then, one can instead pool $v'$ with $v'''$ and $v''$ with $v'''$ into two distinct signals such that the posterior valuation is $E[v|s]$ after either one. In this fashion, the proof of Lemma 1 extends an arbitrary extensionproof information structure that induces price $p$ to a $p$-pairwise information structure that induces the same price and the same payoffs. Extensionproofness of the latter follows from extensionproofness of the former, as the seller could perform the extension herself.

We adapt our notation to $p$-pairwise information structures. Notice that if some signals induce posterior beliefs that have the same support, then by (4) and (5) these posterior beliefs coincide almost surely. All such signals can be merged. We therefore denote signals of a $p$-pairwise information structure directly by $s = (v_L, v_H)$, where $\{v_L, v_H\}$ is the support of $F_s$. For valuations $v < p$, we have almost surely either $v = v_L < v_H$ or $v = v_L = v_H$, that is, the support of $G_v$ is contained in $\{v\} \times (\{v\} \cup [p, 1])$. Define for all $v < p$

$$G_v^H(v_H) := G_v(v, v_H).$$

Similarly, for valuations $v > p$ we have almost surely either $v = v_H > v_L$ or $v = v_H = v_L$, that is, the support of $G_v$ is contained in $([0, p] \cup \{v\}) \times \{v\}$. Define for all $v > p$

$$G_v^L(v_L) := G_v(v_L, v).$$

Hence, for valuations $v < p$ the first component of the signal equals $v$ and the second is drawn from $G_v^H$, and the buyer learns $v$ perfectly with probability $G_v^H(v)$. Similarly, for valuations $v > p$ the first component of the signal is drawn from $G_v^L$ and the second equals $v$, and the buyer learns $v$ perfectly with probability $1 - G_v^L(p)$.

Next, we turn to the seller's response against a $p$-pairwise information structure. For an arbitrary price $q > p$, we construct an extension that maximizes the probability of trade given that price, analogously to Section 4. Regardless of the extension, there will be trade with probability one after all signals $s = (v_L, v_H)$ with $v_L = v_H \geq q$. Moreover, there will be no trade after all $s$ with $v_H < q$. Consider the following extension, performed for all signals $s$ with $v_L < v_H \in [q, 1]$: If $v = v_H$, display a signal $BUY$ with probability one. If $v = v_L$, display $BUY$ with probability

$$x_q(v_L, v_H) := \frac{p - v_L}{v_H - p} \frac{v_H - q}{q - v_L}.$$

To see that this extension maximizes the probability of trade at price $q$, note that given (5) the posterior valuation upon observing $s$ and $BUY$ is exactly $q$:

$$E[v|s, BUY] = \frac{F_s(v_L)x_q(v_L, v_H)v_L + [1 - F_s(v_L)]v_H}{F_s(v_L)x_q(v_L, v_H) + 1 - F_s(v_L)} = q.$$

Consequently, for any $s = (v_L, v_H)$ with $v_L < v_H \in [q, 1]$, the extension persuades the buyer to buy with probability one if $v = v_H$ and with the highest possible probability if $v = v_L$. We call this extension *q-optimal.* The following lemma summarizes.

**Lemma 2.** *Under a q-optimal extension of a p-pairwise information structure, the probability of trade conditional on the true valuation $v$ and the signal $s = (v_L, v_H)$ is one if $v = v_H \geq q$, $x_q(v_L, v_H)$ if $v = v_L < v_H \in [q, 1]$, and zero otherwise.*

We now consider the problem of designing a $p$-pairwise information structure that minimizes the seller's gain from an arbitrary $q$-optimal extensions while inducing a given combination of buyer payoff and price. We will ultimately state this problem as an optimal transport problem, where the choice set is a set of all bivariate distribution functions with given marginals.

First, we establish an equivalence between $p$-pairwise information structures and certain bivariate distribution functions. A distribution function $J$ on $[0, p] \times [p, 1]$ is *p-pairwise* if its marginals are

$$\begin{aligned} J^L(v_L) := J(v_L, 1) &= \frac{1}{c} \int_0^{v_L} \alpha(v)(p - v)dF(v), \\ J^H(v_H) := J(p, v_H) &= \frac{1}{c} \int_p^{v_H} \alpha(v)(v - p)dF(v), \end{aligned}$$

where $c > 0$ is a parameter and $\alpha$ a function from $[0, 1]$ to $[0, 1]$ such that

$$c = \int_0^p \alpha(v)(p - v)dF(v) = \int_p^1 \alpha(v)(v - p)dF(v).$$

A $p$-pairwise information structure $(S,(G_v))$ and a $p$-pairwise distribution function $J$ are *equivalent* if[9]

$$J(v_L,v_H)=\frac{1}{c}\int_0^{v_L}\int_p^{v_H} dG_v^H(u)(p-v)dF(v) \quad \text{and} \quad \alpha(v)=\begin{cases}1-G_v^H(v) & \text{for } v<p,\\ G_v^L(p) & \text{for } v>p.\end{cases}$$

**Lemma 3.** *Every p-pairwise information structure that is not perfect is equivalent to a unique p-pairwise distribution function. Every p-pairwise distribution function is equivalent to an almost everywhere unique p-pairwise information structure.*

Under a $p$-pairwise information structure, each valuation $v$ is pooled into posterior valuation $p$ with some probability $\alpha(v)$ and is perfectly disclosed with probability $1-\alpha(v)$. If the buyer updates to posterior valuation $p$ and buys at price $p$, he makes a loss whenever his true valuation is smaller than $p$ and a profit whenever his true valuation is greater. The parameter $c$ measures both the aggregate loss of the valuations below $p$ and the aggregate profit of the valuations above $p$, which are equal because the posterior valuation is $p$. The marginal $J^L$ gives for each $v_L\in[0,p]$ the (share of the) aggregate loss that valuations $v\leq v_L$ contribute, and the marginal $J^H$ gives for each $v_H\in[p,1]$ the (share of the) aggregate profit that valuations $v\in[p,v_H]$ contribute.

Now, the bivariate distribution $J$ describes how the (shares of) profits and losses are matched with each other, or, put differently, how the loss mass from $J^L$ is transported to $J^H$.[10] For every $v_H\in[p,1]$, the marginal $J^H$ specifies how much loss mass must be matched with the profit mass of the valuations $v\in[p,v_H]$—so much that for almost every pair $(v_L,v_H)$, the matched loss from $v_L$ exactly balances the matched profit from $v_H$. This condition is equivalent to requiring that for almost every signal $s=(v_L,v_H)$, the posterior valuation is exactly $p$, which explains the equivalence between $p$-pairwise distribution functions and $p$-pairwise information structures. Indeed, when choosing among $p$-pairwise information structures, fixing the probabilty $\alpha(v)$ with which each

[9]As $c>0$ by definition, there is no $J$ that is equivalent to a degenerate $p$-pairwise $(S,(G_v))$ where $1-G_v^H(v)=0$ for almost all $v<p$ and $G_v^L(p)=0$ for almost all $v>p$, which corresponds to perfect information. Perfect information is irrelevant for the following analysis that uses $J$.

[10]We can also say that $J$ describes how the profit mass from $J^H$ is transported to $J^L$.

valuation $v$ is pooled amounts to fixing the marginals $J^L$ and $J^H$, and fixing how the valuations are pooled amounts to fixing the distribution $J$ beyond the marginals.

By Lemma 3, a $p$-pairwise information structure and the equivalent $p$-pairwise distribution function are interchangeable. Consider a $p$-pairwise distribution function $J$ that induces price $p$. Observe that the induced buyer payoff is

$$\int_p^1 (v-p)dF(v) - c \tag{6}$$

and the induced seller payoff is

$$\left[1 - F(p) + \int_0^p \alpha(v)dF(v)\right] p. \tag{7}$$

We use $J$ to quantify the probability of trade under a $q$-optimal extension. Define

$$\phi_q(v_L, v_H) := \max\left\{\frac{(v_H - q)c}{(v_H - p)(q - v_L)}, 0\right\}.$$

Then, according to Lemma 2, the probability of trade given price $q$ is

$$\begin{aligned}
&\int_0^p \int_p^1 \max\{x_q(v_L, v_H), 0\} dG^H_{v_L}(v_H)dF(v_L) + 1 - F(q) \\
&\quad = \int_0^p \int_p^1 \phi_q(v_L, v_H)\frac{1}{c}(p - v_L)dG^H_{v_L}(v_H)dF(v_L) + 1 - F(q) \\
&\quad = \int_S \phi_q(v_L, v_H)dJ(v_L, v_H) + 1 - F(q).
\end{aligned} \tag{8}$$

Thus, using $J$ the probability of trade under a $q$-optimal extension can be expressed as an expectation of the function $\phi_q$. Informally speaking, $\phi_q(v_L, v_H)$ is the amount of the prior density $f(v_L)$ that can be pooled into posterior valuation $q$ per unit of matched loss and profit between $v_L$ and $v_H$. Importantly, the function $\phi_q$ is supermodular.

It turns out that we can focus on $p$-pairwise distribution functions $J$ under which each valuation is either always or never pooled into posterior valuation $p$ (i.e., $\alpha(v) \in \{0, 1\}$ for all $v$) and under which those valuations that are pooled constitute and interval. Observe that if the interval $[\underline{v}, \overline{v}]$ is pooled into $p$, then $\int_{\underline{v}}^{\overline{v}} (v-p)dF(v) = 0$, and so $\overline{v}$ is uniquely determined by $p$ and $\underline{v}$. A $p$-pairwise distribution function $J$ will be called $(p, \underline{v})$-*pairwise* if for the corresponding $\overline{v}$,

$$\alpha(v) = \begin{cases} 1 & \text{for } v \in [\underline{v}, \overline{v}], \\ 0 & \text{for } v \notin [\underline{v}, \overline{v}]. \end{cases}$$

**Lemma 4.** *For every extensionproof p-pairwise distribution function that induces price $p$, there exists an extensionproof $(p, \underline{v})$-pairwise distribution function that induces the same price, the same buyer payoff, and a weakly higher seller payoff.*

Now, fix $p \in [0,1]$ and $\underline{v} < p$ such that the $(p, \underline{v})$-pairwise distribution functions induce price $p$. This also fixes the buyer and the seller payoff. What is not yet fixed is, informally, how the valuations $v_L \in [\underline{v}, p)$ are pairwisely pooled with the valuations $v_H \in (p, \overline{v}]$, possibly in a stochastic way, such that the posterior valuation is always $p$. Consider the problem of fixing this pairwise pooling so that it is "as extensionproof" as possible. That is, consider the problem of choosing among the $(p, \underline{v})$-pairwise distribution functions to minimize the probability of trade (8) under an arbitrary $q$-optimal extension:

$$\min_J \quad \int_S \phi_q(v_L, v_H) dJ(v_L, v_H)$$
$$\text{s.t.} \quad J^L(v_L) = \frac{1}{c}\int_{\underline{v}}^{v_L}(p - v)dF(v) \text{ and } J^H(v_H) = \frac{1}{c}\int_p^{v_H}(v - p)dF(v).$$

This is an optimal-transport problem. By the supermodularity of $\phi_q$, the problem is solved by the Fréchet-Hoeffding lower bound

$$\underline{J}(v_L, v_H) := \max\{J^L(v_L) + J^H(v_H) - 1, 0\}$$

(see, e.g., Marshall, Olkin, and Arnold, 2011, Corollary 12.M.3.a). As $\underline{J}$ is independent of $q$, it simultaneously minimizes the seller's gain from every $q$-optimal extension, rendering $\underline{J}$ extensionproof whenever extensionproof $(p, \underline{v})$-pairwise distribution functions exist.

The support of $\underline{J}$ consists of all pairs $(v_L, v_H)$ such that $J^L(v_L) + J^H(v_H) - 1 = 0$, which can be written as $\int_{v_L}^{v_H}(p - v)dF(v) = 0$. Hence, the equivalent $p$-pairwise information structure is constructed as follows: If $v \notin [\underline{v}, \overline{v}]$, display $s = (v, v)$. If $v \in [\underline{v}, \overline{v}]$, display the signal $s = (v_L, v_H) \in [\underline{v}, p] \times [p, \overline{v}]$ that (uniquely) solves

$$v \in \{v_L, v_H\} \quad \text{and} \quad \int_{v_L}^{v_H}(p - v)dF(v) = 0.$$

We call this the $(p, \underline{v})$-*negative-assortative* information structure.

Consider any extensionproof information structure that is not perfect and induces price $p$. By Lemma 1, there is an extensionproof $p$-pairwise information structure with equivalent distribution function $J$ (Lemma 3) that induces the same price and payoffs. Then, by Lemma 4, there is an extensionproof $(p, \underline{v})$-pairwise $J$ that weakly increases the induced seller payoff and that, as we have just shown, is equivalent to the $(p, \underline{v})$-negative-assortative information structure. We have established our main result.[11]

**Theorem 1.** *For every extensionproof information structure that induces price $p$, there exists an extensionproof $(p, \underline{v})$-negative-assortative information structure that induces the same price, the same buyer payoff, and a weakly higher seller payoff.*

According to this result, negative assortative information structures implement every implementable price and buyer payoff and the highest possible corresponding seller payoff. Hence, attention can be restricted to this class of information structures whenever the designer's objective is increasing in the buyer payoff and the seller payoff.

# 6 Buyer-Optimal Information Structures

By Theorem 1, the search for buyer-optimal information structures can be restricted to $(p, \underline{v})$-negative-assortative information structures, that is, to choosing the two parameters $p$ and $\underline{v}$. Before stating this problem, note that the pairs of valuations that are pooled under a $(p, \underline{v})$-negative-assortative information structure are determined by the strictly decreasing function $\mu_p\colon [\underline{v}, \overline{v}] \to [\underline{v}, \overline{v}]$ that is implicitly defined by

$$\mu_p(v) \neq v \quad \text{and} \quad \int_v^{\mu_p(v)} (p-u)dF(u) = 0 \tag{9}$$

for $v \neq p$ and by $\mu_p(p) = p$. Thus, $v$ is pooled with $\mu_p(v)$. In particular, $\mu_p(\mu_p(v)) = v$ and $\mu_p(\underline{v}) = \overline{v}$.

A $(p, \underline{v})$-negative-assortative information structure yields buyer payoff $\int_{\underline{v}}^1 (v-p)dF(v)$ and seller payoff $[1-F(\underline{v})]p$ if it induces price $p$ and is extensionproof. The information structure induces price $p$ if

$$[1-F(\underline{v})]p \geq [1-F(q)]q \quad \text{for all } q \notin (p, \mu_p(\underline{v})). \tag{10}$$

[11] Note that perfect information is accommodated by setting $\underline{v} = p$.

As the seller can always extend to perfect information, she must at least get her perfect-information payoff $\Pi^* = \max_q [1 - F(q)]q$. Accordingly, we can replace (10) by the simpler condition

$$[1 - F(\underline{v})]p \geq \Pi^*. \tag{11}$$

Let the seller's payoff from charging price $q \in (p, \mu_p(\underline{v}))$ and performing the $q$-optimal extension be defined as

$$\Psi(q, p, \underline{v}) := \left[1 - F(q) + \int_{\underline{v}}^{\mu_p(q)} x_q(v, \mu_p(v))dF(v)\right] q.$$

The information structure is extensionproof if the seller payoff at price $p$ is greater than her payoff from any extension,

$$[1 - F(\underline{v})]p \geq \Psi(q, p, \underline{v}) \quad \text{for all } q \in (p, \mu_p(\underline{v})). \tag{12}$$

Thus, the problem of designing a buyer-optimal $(p, \underline{v})$-negative-assortative information structure can be stated as

$$\max_{(p,\underline{v})} \int_{\underline{v}}^{1} (v - p)dF(v) \quad \text{s.t. (11) and (12).} \tag{13}$$

We refer to solutions of problem (13) as *optimal* $(p, \underline{v})$. The following proposition characterizes these solutions.

**Proposition 2.** *Optimal* $(p, \underline{v})$ *exist and have the following properties:*

*(i) If* $(p, \underline{v}) = (\Pi^*, 0)$ *satisfies* (12)*, then* $(p, \underline{v}) = (\Pi^*, 0)$ *is uniquely optimal and implements buyer payoff* $\overline{U} := E[v] - \Pi^*$*.*

*(ii) If* $(p, \underline{v}) = (\Pi^*, 0)$ *does not satisfy* (12)*, then there exist*

$$\omega := \min\{\, \underline{v} : (p, \underline{v}) = (\Pi^*/[1 - F(\underline{v})], \underline{v}) \text{ satisfies } (12) \,\}$$
$$\text{and} \quad \hat{p}(\underline{v}) := \min\{\, p : (p, \underline{v}) \text{ satisfies (11) and (12)} \,\} \quad \text{for all } \underline{v} \in [0, \omega],$$

*where* $\omega > 0$*,* $\hat{p}(0) > \Pi^*$*, and* $\hat{p}(\cdot)$ *is strictly increasing. If* $(p, \underline{v})$ *is optimal, then* $\underline{v} \in [0, \omega]$ *and* $p = \hat{p}(\underline{v})$*. The implemented buyer payoff is strictly less than* $\overline{U}$*.*

As the seller payoff is at least $\Pi^*$ and the social surplus at most $E[v]$, which is obtained when trade happens with probability one, the buyer payoff is at most $\overline{U} = E[v] - \Pi^*$ (like in the uniform case in Section 4). Among $(p, \underline{v})$-negative-assortative information structures, only $(p, \underline{v}) = (\Pi^*, 0)$ induces both trade with probability one and seller payoff $\Pi^*$, making it the unique candidate for attaining the upper bound $\overline{U}$ on the buyer payoff. Hence, if $(p, \underline{v}) = (\Pi^*, 0)$ satisfies the extensionproofness constraint (12), this is the unique optimum and the resulting buyer payoff is $\overline{U}$, which is case (i) of Proposition 2.

If $(p, \underline{v}) = (\Pi^*, 0)$ does not satisfy (12), the buyer payoff cannot attain $\overline{U}$, and case (ii) of Proposition 2 applies. Any $(p, \underline{v})$ that satisfies (11) and (12) then either induces inefficient trade ($\underline{v} > 0$) or a seller payoff strictly greater than under perfect information ($[1 - F(\underline{v})]p > \Pi^*$) or both. With the value $\omega$, the proposition provides an upper bound on the inefficiency that optimal $(p, \underline{v})$ may introduce: $\omega$ is the smallest $\underline{v}$ among all feasible $(p, \underline{v})$ that induce seller payoff $\Pi^*$. Any $\underline{v} > \omega$ is dominated because it yields less social surplus but a weakly greater seller payoff. Indeed, the proposition transforms problem (13) into a one-dimensional problem with $\underline{v} \in [0, \omega]$ as the only choice variable. Specifically, solutions to (13) take the form $(\hat{p}(\underline{v}), \underline{v})$, where $\hat{p}(\underline{v})$ is the lowest price $p$ that renders $(p, \underline{v})$ feasible. The function $\hat{p}$ is strictly increasing. Hence, an optimal choice of $\underline{v}$ resolves a trade-off between lower prices $\hat{p}(\underline{v})$ and more information (higher $\underline{v}$).

Which of the two cases of Proposition 2 obtains, and how the trade-off in the latter case is resolved, depends on the prior $F$. In what follows, we focus on a class of priors where the extensionproofness constraint (12) is particularly well behaved.

Define the prior $F$ to be *regular* if $[1 - F(v)]v$ is strictly quasiconcave, the PDF $f$ is twice differentiable for all $v \in (0, 1)$, and

$$f''(v)v \geq -2f'(v) + f'(v) \max\left\{0, \frac{f'(v)v}{12f(v)} - \frac{1}{4}\right\} \quad \text{for all } v \in (\Pi^*, 1). \tag{14}$$

The following example presents a class of regular priors.[12]

**Example 1.** *Suppose the prior is a beta distribution with parameters* $a \geq 1$ *and* $b \in (0, 1]$.

[12] While $f' \geq 0$ for all priors in this class, regularity does not preclude $f' < 0$. For example, a prior with $F(v) = 1 + \ln(v)/e$ for $v \geq e^{-2}$, $F(v) \geq 1 - f(v)v$ for $v < e^{-2}$, and twice differentiable $f$ is regular.

*The corresponding PDF is*

$$f(v) = \frac{v^{a-1}(1-v)^{b-1}}{\int_0^1 u^{a-1}(1-u)^{b-1}du}.$$

*Since* $f'(v) = [(a-1)v^{-1} + (1-b)(1-v)^{-1}]f(v) \geq 0$, $[1-F(p)]p$ *is strictly concave. Moreover,* (14) *holds if*

$$f''(v)v \geq -2f'(v) + \frac{[f'(v)]^2 v}{f(v)}.$$

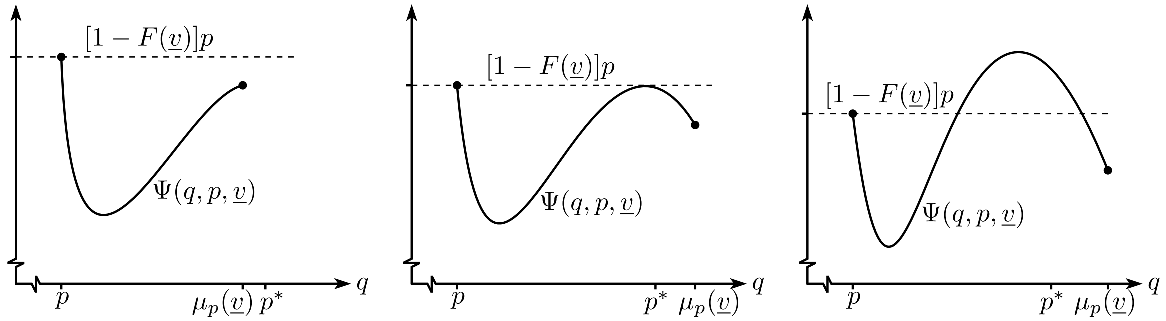
*Using* $f''(v) = \{[(a-1)v^{-1} + (1-b)(1-v)^{-1}]^2 - (a-1)v^{-2} + (1-b)(1-v)^{-2}\}f(v)$, *this inequality simplifies to* $(1-b)(1-v)^{-2}v \geq -(a-1)v^{-1} - 2(1-b)(1-v)^{-1}$, *which clearly holds. Thus, the prior is regular.*

Suppose the prior $F$ is regular. Recall from Section 3 that the price $p^*$ maximizes the seller payoff under perfect information. The strict quasiconcavity of $[1-F(q)]q$ implies that $[1-F(q)]q$ is strictly increasing and decreasing, respectively, at all prices $q$ below and above $p^*$. Condition (14), in turn, implies that the third derivative of the seller's extension payoff $\Psi(\cdot, p, \underline{v})$ is negative, as stated in the following lemma together with properties of the first derivative.

**Lemma 5.** *Suppose the prior $F$ is regular and* (11) *holds. Then for all* $q \in (p, \mu_p(\underline{v}))$, $\Psi(q, p, \underline{v})$ *is thrice differentiable with respect to $q$ and* $\partial^3\Psi(q, p, \underline{v})/\partial q^3 \leq 0$. *Moreover,*

$$\lim_{q \downarrow p} \frac{\partial \Psi(q, p, \underline{v})}{\partial q} = -\infty \qquad \textit{and} \qquad \left.\frac{\partial \Psi(q, p, \underline{v})}{\partial q}\right|_{q=\mu_p(\underline{v})} = \left.\frac{\partial [1-F(q)]q}{\partial q}\right|_{q=\mu_p(\underline{v})}.$$

Since the seller's extension payoff $\Psi(\cdot, p, \underline{v})$ as a function of $q$ is decreasing and infinitely steep at the lower limit of its domain $(p, \mu_p(\underline{v}))$, it is convex for low values of $q$. Hence, the negative third derivative implies that $\Psi(\cdot, p, \underline{v})$ is either convex throughout or switches to concave for high values of $q$. Figure 1 depicts three curves with this property. Because the first derivative of $\Psi(\cdot, p, \underline{v})$ equals that of $[1-F(q)]q$ at the upper limit of its domain $(p, \mu_p(\underline{v}))$, $\Psi(\cdot, p, \underline{v})$ is increasing at $q = \mu_p(\underline{v})$ if and only if $\mu_p(\underline{v}) \leq p^*$. Thus, the extension payoff is quasiconvex if $\mu_p(\underline{v}) \leq p^*$, as depicted in panel (a) of Figure 1. In that case, any $(p, \underline{v})$ that satisfies (11) also satisfies the extensionproofness constraint (12). If $\mu_p(\underline{v}) > p^*$, by contrast, then $(p, \underline{v})$ may or may not be extensionproof, as panels (b) and (c) of Figure 1 illustrate.




(a) $(p,\underline{v})$ is extensionproof (b) $(p,\underline{v})$ is extensionproof (c) $(p,\underline{v})$ is not extensionproof

Figure 1: Possible shapes of the seller's extension payoff $\Psi(\cdot,p,\underline{v})$ under regularity

Now, consider $(p,\underline{v}) = (\Pi^*,0)$. As we just found, if $\mu_{\Pi^*}(0) \le p^*$, then $(\Pi^*,0)$ is extensionproof. If $\mu_{\Pi^*}(0) > p^*$, then the seller can perform the $p^*$-optimal extension and sell at $p^*$ with strictly greater probability than under perfect information, which means $\Psi(p^*,\Pi^*,0) > \Pi^*$, that is, $(\Pi^*,0)$ is not extensionsproof. Consequently, whether case (i) or (ii) of Proposition 2 applies only depends on whether or not $\mu_{\Pi^*}(0) \le p^*$. Moreover, according to the definition of $\mu_p$ in (9), $\mu_{\Pi^*}(0) \le p^*$ is equivalent to $E[v|v \le p^*] \ge \Pi^*$. The following proposition incorporates this insight and further strengthens our characterization of optimal $(p,\underline{v})$ for regular priors.

**Proposition 3.** *Suppose the prior $F$ is regular.*

*(i) If $E[v|v \le p^*] \ge \Pi^*$, then $(p,\underline{v}) = (\Pi^*,0)$ is uniquely optimal and implements buyer payoff $\overline{U}$ and seller payoff $\Pi^*$.*

*(ii) If $E[v|v \le p^*] < \Pi^*$, then optimal $(p,\underline{v})$ implement a buyer payoff strictly less than $\overline{U}$ and a seller payoff strictly greater than $\Pi^*$. Moreover,*

$$\omega = \min\{\,\underline{v} : [1-F(\underline{v})]E[v|\underline{v} \le v \le p^*] = \Pi^*\,\} < p^*$$

*and the implemented buyer payoff is strictly greater than $\int_\omega^1 v dF(v) - \Pi^* > \int_{p^*}^1 (v-p^*)dF(v)$.*

First, Proposition 3 provides a simple condition to determine whether or not the buyer payoff attains $\overline{U}$: this depends on whether or not under perfect information, the buyer's valuation conditional on not buying, $E[v|v \le p^*]$, is greater than the seller payoff.

Second, whenever the buyer payoff does not attain $\overline{U}$, the seller obtains a payoff strictly greater than $\Pi^*$. That is, the seller strictly prefers buyer-optimal information over perfect information. To prove this result, we show in the Appendix that, for $\underline{v}$ sufficiently close to $\omega$, increasing $\underline{v}$ to $\omega$ leads to a relatively large increase in the price $\hat{p}(\underline{v})$ and thus to a decrease in the buyer payoff from $(\hat{p}(\underline{v}), \underline{v})$. Hence $(\hat{p}(\omega), \omega)$ is not optimal, and since among all $\underline{v} \in [0, \omega]$ only $\underline{v} = \omega$ implements seller payoff $\Pi^*$, the optimal $(\hat{p}(\underline{v}), \underline{v})$ implement a strictly higher seller payoff. The characterization of the function $\hat{p}$ contained in the proof uses that for any $(\hat{p}(\underline{v}), \underline{v})$ the extensionproofness constraint (12) binds at just one $q$, which satisfies a first-order condition. This is illustrated in panel (b) of Figure 1 and is, as we show, yet another implication of regularity.

Finally, Proposition 3 provides a simple definition of $\omega$ and improves the lower bound on the buyer payoff we established in Section 3: the buyer payoff is bounded away from the perfect-information payoff $\int_{p^*}^1 (v - p^*) dF(v)$.

We now present two classes of regular priors that are special cases of Example 1. In the first example, the buyer payoff attains $\overline{U}$.

**Example 2.** *Let the prior be a beta distribution with parameters $a \geq 1$ and $b = 1$. Then, $F(v) = v^a$, $p^* = (1 + a)^{-1/a}$ and $E[v|v \leq p^*] = p^* a/(1 + a) = \Pi^*$. Hence, case (i) of Proposition 3 applies. For $a = 1$, we obtain the uniform prior studied in Section 4.*

In the second example, case (ii) of Proposition 3 applies. Also with a specific regular prior, optimal $(p, \underline{v})$ typically cannot be obtained in closed form.[13] Yet, the problem lends itself to numerical simulation. While according to the proposition $\underline{v} = \omega$ is not optimal, we ran simulations showing that $\underline{v} = 0$ is also not optimal. Hence, buyer-optimal information structures can result in trade with probability strictly less than one, that is, in an inefficient allocation.

**Example 3.** *Let the prior be a beta distribution with parameters $a = 1$ and $b \in (0, 1)$. Then, $F(v) = 1 - (1 - v)^b$, $p^* = (1 + b)^{-1}$, and one can show that $E[v|v \leq p^*] < \Pi^*$. For example, for $b = 1/2$, $E[v|v \leq p^*] = (6\sqrt{3} - 8)/(9\sqrt{3} - 9) < 2/(3\sqrt{3}) = \Pi^*$. Hence,*

[13] In particular, the pooled valuation $\mu_p(v)$ typically cannot be obtained in closed form. E.g., if $F$ is a polynomial of degree $m$, one can show that $\mu_p(v)$ is a root of a polynomial of degree $m + 1$.

| | $b$ | 0.05 | 0.10 | 0.20 | 0.25 | 0.50 | 0.75 |
|---|---|---|---|---|---|---|---|
| $\underline{v} = 0$ | $\hat{p}(0)$ | 0.82549 | 0.72264 | 0.58724 | 0.53881 | 0.38586 | 0.30282 |
| | buyer payoff | 0.12689 | 0.18646 | 0.24609 | 0.26119 | 0.28081 | 0.26860 |
| $\underline{v} = \underline{v}_{\text{opt}}$ | $\underline{v}_{\text{opt}}$ | 0.17719 | 0.14438 | 0.09145 | 0.07396 | 0.02775 | 0.00917 |
| | $\hat{p}(\underline{v}_{\text{opt}})$ | 0.83178 | 0.73151 | 0.59658 | 0.54758 | 0.39085 | 0.30485 |
| | buyer payoff | 0.12778 | 0.18775 | 0.24721 | 0.26212 | 0.28109 | 0.26865 |

Table 1: Simulation results for Example 3

*case (ii) of Proposition 3 applies. Table 1 presents the results of numerical simulations for six different values for b.*[14] *We found that the buyer payoff under* $(\hat{p}(\underline{v}), \underline{v})$ *is strictly quasiconcave in* $\underline{v}$ *and maximized at the value* $\underline{v} = \underline{v}_{\text{opt}}$ *that is given in the table along with the corresponding optimal price* $\hat{p}(\underline{v}_{\text{opt}})$ *and buyer payoff. In all six cases,* $\underline{v}_{\text{opt}}$ *is strictly positive. Table 1 also contains the price and buyer payoff under* $(\hat{p}(0), 0)$*, which maximizes the buyer payoff under efficient trade and is strictly suboptimal.*

In this section, we have focused on $(p, \underline{v})$-negative-assortative information structures. Using Theorem 1, our findings concerning the buyer payoff under optimal $(p, \underline{v})$ immediately translate to buyer-optimal information structures in general. Regarding the seller payoff, however, Theorem 1 leaves open the possibility that under a regular prior where $E[v|v \leq p^*] < \Pi^*$, there is a buyer-optimal information structure that just yields seller payoff $\Pi^*$. We prove that this cannot be the case as part of the following corollary, which summarizes the main results for general information structures.

**Corollary 1.** *Buyer-optimal information structures implement buyer payoff* $\overline{U}$ *if* $(p, \underline{v}) = (\Pi^*, 0)$ *satisfies* (12) *and strictly less otherwise. For a regular prior, buyer-optimal information structures implement buyer payoff* $\overline{U}$ *and seller payoff* $\Pi^*$ *if* $E[v|v \leq p^*] \geq \Pi^*$ *and a buyer payoff strictly less than* $\overline{U}$ *and a seller payoff strictly greater than* $\Pi^*$ *otherwise.*

[14]The *Mathematica* source code for these simulations is available on request from the authors.

# 7 Discussion

In this section, we consider a different timing that allows for sequential screening, study a weaker extensionproofness constraint, investigate how the seller's ability to add information changes the design problem, and analyze a variant of the model where the seller chooses the information structure and the buyer can extend it.

## 7.1 Sequential Screening

In the original model, the buyer obtains all information at the same time. A standard argument shows that it is then optimal for the seller to make a take-it-or-leave-it price offer (see Riley and Zeckhauser, 1983). We assume now that any additional information the seller discloses is observed by the buyer at a later point in time. Moreover, the seller can enter into a contract with the buyer *after* the buyer obtains information from the original information structure, but *before* the seller releases any additional information. As a consequence, the seller can screen the buyer sequentially if she discloses more.

The new timing is as follows. First, the buyer (or a regulator) chooses an information structure $(S^a, (G^a_v))$. He now observes the signal $s^a$ immediately. Then the seller, knowing $(S^a, (G^a_v))$ but not $s^a$, offers a mechanism that determines both the additional information to be released and the terms of trade. The buyer must accept or reject the mechanism before obtaining the additional information.

We continue to assume that by releasing additional information, the seller may extend $(S^a, (G^a_v))$ to any information structure $(S^a \times S^b, (G_v))$ for some $S^b$ as defined in Section 2. Equivalent to this, we say that the seller may provide any extension, denoted by $X := (S^b, (G^b_{v,s^a}))$, where $G^b_{v,s^a}(s^b)$ is such that $G_v(s^a, s^b) = \int_{\{e \in S^a : e \leq s^a\}} G^b_{v,e}(s^b) dG^a_v(e)$. That is, $X$ is a combination of a set of signals and CDFs under which a signal $s^b \in S^b$ is drawn conditional on the valuation $v$ and the original signal $s^a$.

We focus on deterministic mechanisms represented by menus of the form

$$\{(X(\tilde{s}^a), c(\tilde{s}^a), p(\tilde{s}^a)) : \tilde{s}^a \in S^a\}.$$

If the buyer accepts, he chooses a combination $(X(\tilde{s}^a), c(\tilde{s}^a), p(\tilde{s}^a))$ from the menu, where $X(\tilde{s}^a)$ is an extension and $c(\tilde{s}^a), p(\tilde{s}^a)$ represent a call option. Upon making his choice as indicated by $\tilde{s}^a$, the buyer observes an additional signal $s^b$ from $X(\tilde{s}^a)$, pays the option price $c(\tilde{s}^a)$ in any case, and may purchase the object at the strike price $p(\tilde{s}^a)$. We assume that the buyer purchases the object if and only if $E[v|s^a, s^b] \geq p(\tilde{s}^a)$.

A menu is *incentive compatible* if the buyer's expected payoff conditional on the signal $s^a$ from the original information structure is maximized when he chooses the combination $(X(s^a), c(s^a), p(s^a))$. Invoking the revelation principle, we may restrict attention to incentive-compatible menus. A menu is *individually rational* if the buyer's expected payoff conditional on $s^a$ is greater than or equal to zero when he chooses $(X(s^a), c(s^a), p(s^a))$.

*Posted-price mechanisms* are a particular class of menus where for some extension $X$ and some strike price $p$, $(X(\tilde{s}^a), c(\tilde{s}^a), p(\tilde{s}^a)) = (X, 0, p)$ for all $\tilde{s}^a \in S^a$. Such a menu is equivalent to just one combination, with an option price of zero. Thus, the buyer is not screened sequentially: his only decision is whether or not to buy the object at price $p$, and he obtains all information beforehand. Note that every posted-price mechanism is automatically incentive compatible and individually rational.

In the original model, the seller was restricted to using posted-price mechanisms. As we show now, the seller does not gain from using menus other than posted-price mechanisms also under the new timing, where she could screen the buyer sequentially. Moreover, there are no such other menus that are optimally chosen by the seller and yield a higher buyer payoff. Hence, our analysis of buyer-optimal information structures in the preceding sections carries over without change.

**Proposition 4.** *Let $(S^a, (G^a_v))$ be any information structure. For every incentive-compatible and individually rational menu, there exists a posted-price mechanism that yields the same buyer payoff and a weakly higher seller payoff.*

To build intuition for this result, suppose the additional information arrives exogenously. Ideally, the seller would just post a take-it-or-leave-it price offer equal to the posterior valuation conditional on the signal from the original information structure. Since only the buyer knows the signal, this scheme is not possible. Usually, it is then advantageous to screen the signal and offer a menu of call options (see, e.g., Courty and

Li, 2000). In our setting, the seller has another tool that allows making the probability of trade contingent on the signal: she can design the additional information. By disclosing more or less, conditional on the true signal, she can change the probability that the buyer buys the object at any given price. This tool is so effective that the seller abstains from screening the signal altogether.[15]

In a seminal paper, Eső and Szentes (2007) study the problem of designing an optimal sequential selling mechanism that determines both the terms of trade and the disclosure of additional information. In their model, the original information is exogenous, and the disclosure cannot condition on it directly. They transform the additional information into a variable that is independent of the original information and find that, under certain distributional assumptions, the optimal mechanism perfectly discloses the variable. In our model, where the original information is endogenous, these distributional assumptions are violated in general. Indeed, take any extensionproof information structure. By Proposition 4, the seller does not gain by disclosing additional information using a (deterministic) sequential mechanism—even if the disclosure *can* condition on the original information directly.[16]

## 7.2 Weak Extensionproofness

So far, we have assumed that the seller's extension can condition on the signal that the buyer receives from the original information structure. Such extensions may be the appropriate notion when, for example, the disclosure concerns different product attributes as described in the Introduction. In other applications, the seller's choice

[15]Smolin (2019) considers a model in which a buyer has private information about the relevance of various product attributes and the seller can disclose information about each attribute. Effectively, the information content of the disclosure for the buyer's valuation then depends on the true attribute relevance. Under certain assumptions, a posted-price offer is optimal for the seller, like in our model.

[16]Krähmer (2018) considers a model with exogenous original information in which a seller can secretly randomize over disclosures. He shows that such randomization is useful for belief elicitation: if the posterior beliefs conditional on the original information have full support, and assuming finitely many posterior beliefs and valuations, the seller can fully extract the social surplus.

of extensions may be more limited (after all, the buyer privately observes the original signal). We now consider the case that the extension can only condition on the valuation.

Formally, if $(S^a, (G_v^a))$ is the original information structure and $(S^a \times S^b, (G_v))$ the extended one, then the extension is *independent* if $G_v = G_v^a G_v^b$ for some CDF $G_v^b$ over $S^b$. An information structure is *weakly extensionproof* if the seller has no incentive to independently extend it. Every extensionproof information structure is, of course, also weakly extensionproof. In particular, a negative assortative information structure is weakly extensionproof if *and only if* it is extensionproof: as under such information structures the signal is a deterministic function of the valuation, nothing changes when the extension cannot condition on the signal directly.

However, Theorem 1, which allowed us to focus on negative assortative information structures, characterizes the implementable buyer payoffs under the original extensionproofness constraint. In the derivation, we used extensions that condition on the original signal to show that the restriction to $p$-pairwise information structures is without loss of generality (Lemma 1) and for the $q$-optimal extension (Lemma 2). This raises the question of whether certain payoffs are implementable by weakly extensionproof information structures but not by extensionproof ones. According to the following proposition, such information structures cannot be $p$-pairwise.

**Proposition 5.** *A $p$-pairwise information structure is extensionproof if and only if it is weakly extensionproof.*

Under the original extensionproofness constraint, the buyer payoff attains the upper bound $\overline{U}$ for some but not all priors. We show now that this remains true under weak extensionproofness and that, for a regular prior, $\overline{U}$ is attained under weak extensionproofness if and only if $\overline{U}$ is attained under extensionproofness. We first establish an auxiliary result, which says that valuations above $p^*$ must not be pooled with valuations below $p^*$. Otherwise, the seller could trade at price $p^*$ with a greater probability than under perfect information, obtaining a payoff greater than $\Pi^*$.

**Lemma 6.** *An information structure $(S, (G_v))$ that induces buyer payoff $\overline{U}$ is weakly* *extensionproof only if*

$$\int_{\{s\in S:F_s(p^*)\in(0,1)\}} d\bar{G}(s) = 0. \tag{15}$$

Now, in order to induce trade at price $\Pi^*$ with probability one and thus attain the upper bound $\overline{U}$, the lowest posterior valuation must be at least $\Pi^*$. By Lemma 6, this requires that the valuations below $p^*$ can be pooled in such a way that the lowest posterior valuation is at least $\Pi^*$. This is possible only if the prior mean $E[v|v \leq p^*]$ of these valuations is greater than $\Pi^*$.

**Proposition 6.** *Weakly extensionproof information structures that induce buyer payoff $\overline{U}$ exist only if $E[v|v \leq p^*] \geq \Pi^*$.*

For a regular prior, Propositions 3 and 6 imply that $E[v|v \leq p^*] \geq \Pi^*$ is necessary and sufficient for attaining $\overline{U}$ under both weak extensionproofness and extensionproofness.

### 7.3 Comparison: No Extension by the Seller

Here, we compare our results with those of Roesler and Szentes (2017), who study buyer-optimal information structures when the seller cannot disclose more. They identify a class of information structures, from now on called the *RS class*, with the property that for every information structure there exists one in this class that induces the same seller payoff and a weakly higher buyer payoff. When the seller cannot disclose more, the only relevant property of an information structure is the induced CDF of posterior valuations. An information structure is in the RS class if and only if the induced CDF of posterior valuations is

$$H_q^B(w) := \begin{cases} 0 & \text{if } w \in [0,q), \\ 1-\frac{q}{w} & \text{if } w \in [q,B), \\ 1 & \text{if } w \in [B,1] \end{cases}$$

for some $q \in (0,1]$ and some $B \in [q,1]$. Moreover, an information structure that induces $H_q^B$ exists if and only if the prior $F$ is a mean-preserving spread of $H_q^B$. For a given $H_q^B$, the seller is indifferent between all prices $p \in [q,B]$. Hence, her payoff is $q$, and the probability of trade is one if she charges price $q$.

According to Roesler and Szentes (2017, Theorem 1), the buyer-optimal information structures in their setting result in trade with probability one at price

$$p^{RS} := \min\left\{q : \exists B \in [q, 1] \text{ s.t. } F \text{ is a mean-preserving spread of } H_q^B\right\}.$$

It turns out that none of these information structures is (weakly) extensionproof.

**Proposition 7.** $p^{RS} < \Pi^*$. *Hence, the information structures that are buyer optimal when the seller cannot provide an extension are not (weakly) extensionproof.*

The (weak) extensionproofness constraint thus always binds, and the seller's ability to add information always makes her strictly better off and the buyer strictly worse off.

Roesler and Szentes (2017, Lemma 1) show that if some information structure results in seller payoff $q$, then the CDF of posterior valuations is a mean-preserving spread of the corresponding CDF $H_q^B$. In this sense, the information structures in the RS class are least informative. In our setting, by contrast, the goal is to implement the desired CDF of posterior valuations with an information structure that is *as* informative as possible to deter extensions. This suggests that focusing on the RS class may cause a loss of generality when the seller can add information. Indeed, the following result shows that the upper bound $\overline{U}$ on the buyer payoff in our setting is never attainable with this class.

**Proposition 8.** *There is no (weakly) extensionproof information structure in the RS class that induces buyer payoff* $\overline{U}$.

## 7.4 Switched Roles in Information Design

Suppose the *seller* chooses the information structure and the *buyer* can extend it. When he decides on the extension, the buyer knows the information structure but not yet the signal. The seller, in turn, knows the extension when she sets the price. One interpretation of this setting is that the extension is performed by a consumer protection agency that reacts to the seller's disclosure of product information.

We assume that the seller always sets the lowest price that is optimal for her. Adapting our terminology, we say that an information structure *induces* price $p$ (and the corresponding payoffs) if $p = \min \operatorname{argmax}_q q \int_q^1 dH(w)$. An information structure is *buyer* *extensionproof* if the buyer has no incentive to extend it. Similar to our original analysis, we can restrict attention to buyer-extensionproof information structures. The seller payoff must again be at least $\Pi^*$, the payoff the seller can get by providing perfect information. In what follows, we show that the seller payoff is actually equal to $\Pi^*$.

Virtually the same argument as in the original model allows us to confine the analysis to $p$-pairwise information structures (cf. Lemma 1).

**Lemma 7.** *Every buyer-extensionproof information structure that induces price* $p$ *and a seller payoff weakly greater than* $\Pi^*$ *can be extended to a* $p$*-pairwise information structure that induces the same price, the same buyer payoff, and the same seller payoff.*

Consider any $p$-pairwise information structure that induces price $p$ and a seller payoff *strictly* greater than $\Pi^*$. Suppose the buyer chooses an extension that in addition reveals whether or not his valuation is below some cutoff $v' < p$. Thus, he perfectly learns his valuation for signals $s = (v_L, v_H)$ with $v_L < v' < v_H$, whereas he learns nothing new for all other signals. Now, let $v'$ be such that, at price $p$, this extension strictly decreases the probability of trade but the seller still obtains a payoff strictly greater than $\Pi^*$. As the extended information structure is still $p$-pairwise, the seller payoff at prices $q > p$ is still at most $[1 - F(q)]q \leq \Pi^*$. Hence, the seller charges a price $q \leq p$, and the buyer payoff strictly increases, that is, the original information structure is not buyer extensionproof. This establishes the following proposition.

**Proposition 9.** *Every buyer-extensionproof information structure that is optimal for the seller induces seller payoff* $\Pi^*$*.*

Perhaps surprisingly, the seller does not have a first-mover advantage: in the original model, where the buyer chooses the information structure and the seller can extend it, the seller payoff can be strictly greater than $\Pi^*$ (see Proposition 3). Intuitively, choosing an extension gives the buyer more direct control over the seller's pricing decision than designing the information structure.

The buyer may indeed prefer the switched roles. Suppose $E[v|v \leq p^*] \geq \Pi^*$ and consider the $(\Pi^*, 0)$-negative-assortative information structure. This information structure is optimal for the seller because she can secure herself (via price $p^*$) payoff $\Pi^*$ no matter what extension the buyer chooses, and by Proposition 9 no other information structure yields more. Moreover, this information structure induces buyer payoff at the upper bound $\overline{U}$, which is more than under any extension. Hence, buyer payoff $\overline{U}$ can be attained whenever $E[v|v \leq p^*] \geq \Pi^*$. In the original model, by contrast, this condition is sufficient for attaining $\overline{U}$ under a regular prior (see Proposition 3) but not in general.[17]

# 8 Conclusion

An important insight from the theory of information design is that buyers who can choose what information to obtain may benefit from staying imperfectly informed about their willingness to pay. In particular, doing so is beneficial if it reduces the dispersion in their estimated willingness to pay in such a way that the seller offers more favorable terms of trade. Studying buyer-optimal regulation of product information, we have dropped the assumption that one side of the market fully controls the information structure: the regulator usually can only impose minimum disclosure requirements, and the seller is free to disclose more. To prevent (harmful) additional disclosure, the regulator's problem must include the constraint that the information structure be extensionproof. As we have shown, this constraint always binds under buyer-optimal regulation, and the seller profits from her partial control of information at the expense of the buyer. Preemptively removing all uncertainty, however, is virtually never the best way to deal with the constraint, that is, staying imperfectly informed still remains beneficial.

Our main contribution is a two-parameter class of information structures that maximially deter extensions, where one parameter determines the price and the other one the information content. This class always contains a buyer-optimal information structure, which can be found once the prior belief over the buyer's valuation is specified. To attain

[17]To demonstrate that $E[v|v \leq p^*] \geq \Pi^*$ is in general not sufficient for attaining $\overline{U}$ in the original model, suppose $f(v) = 1 + 4v$ if $v \in [0, 1/8)$, $f(v) = 3/4$ if $v \in [1/8, 3/8)$, $f(v) = 3 - 4v$ if $v \in [3/8, 1/2)$, and $f(v) = 1$ if $v \in [1/2, 1]$. For this prior, $E[v|v \leq p^*] = \Pi^* = 1/4$. However, by exploiting the symmetry of the PDF around $v = 1/4$, one can analytically show that $(p, \underline{v}) = (1/4, 0)$ violates (12) for $q = 3/8$. Hence, by Corollary 1, buyer payoff $\overline{U}$ cannot be attained.

a lower price than under perfect information, some interval of valuations are pooled such that the posterior valuation equals the desired price; this way, the probability of trade at that price increases, inducing the seller to charge it. One feature stands out: inside the interval, each valuation is pooled with just one other valuation, lower ones being pooled with higher ones in negative assortative fashion.

The optimality of negative assortative information structures makes precise the intuition that, to deter additional disclosure, the regulator should minimize the seller's control of information by requiring the disclosure of anything that leaves the desired dispersion in the estimated willingness to pay unchanged. In particular, the regulator should require the release of any information that helps assessing the relevance of the information that remains concealed, such as the importance of non-revealed features of the product. Depending on how the buyer assesses the size of the impact that learning also the non-revealed features would have on his evaluation of the product, he will deem more or less extreme valuations likely, just as negative assortative pooling prescribes. In practice, this could mean requiring the publication of test results from which the buyer can learn how strongly his evaluation would change when testing it himself. Or when a set of product attributes are substitutes for each other, some of those attributes should be revealed to the buyer: depending on how much the buyer already likes the revealed attributes, he deems the unrevealed substitutes less or more important.

An ideal solution for the buyer, who gets the residual surplus, would be an (extensionproof) information structure that simultaneously maximizes the social surplus and only grants the lowest implementable payoff to the seller, which equals the seller payoff under perfect information. Often, this ideal is indeed attainable. In particular, buyer-optimal regulation need not come at a cost in terms of social surplus. In other cases, however, it is advantageous to neither maximize the social surplus nor restrict the seller to the lowest implementable payoff. Hence, the seller may strictly prefer buyer-optimal information over perfect information.

Our analysis has applications beyond buyer-optimal information. First, negative assortative information structures implement every implementable price and buyer payoff together with the highest corresponding seller payoff. Hence, attention can also be restricted to such information structures if the designer's objective is an increasing function of buyer and seller payoff, possibly subject to a price constraint. Second, and more generally, we have uncovered a connection between information design and matching, or optimal transport: information design problems that require a given set of states to be optimally pooled pairwisely into a given posterior mean can be transformed into optimal-transport problems. Whenever not just the distribution of posterior means matters, but also how states are pooled into these means, this connection may prove useful.

# A Appendix: Proofs

**Proof of Proposition 1.** We will construct an extensionproof information structure that induces a buyer payoff strictly greater than $\int_{p^*}^1 (v - p^*)dF(v)$. As a starting point, consider the following class of information structures that are parameterized by a cutoff $\underline{v} \leq p^*$: signal $s = v$ is displayed if $v \notin [\underline{v}, p^*]$ whereas the same signal $s = \hat{s}$ is displayed if $v \in [\underline{v}, p^*]$. Hence, after signal $\hat{s}$ the posterior valuation is $E[v|v \in [\underline{v}, p^*]]$ and otherwise the buyer perfectly learns $v$. If the seller sets price $p = E[v|v \in [\underline{v}, p^*]]$, her payoff is

$$\Omega(\underline{v}) := [1 - F(\underline{v})]E[v|v \in [\underline{v}, p^*]].$$

Other prices $p \in [\underline{v}, p^*]$ are clearly suboptimal and prices $p \notin [\underline{v}, p^*]$ result in a payoff of at most $\Pi^*$, as under perfect information. Consequently, the information structure induces price $E[v|v \in [\underline{v}, p^*]]$ if $\Omega(\underline{v}) > \Pi^*$.

We now show that for some $\underline{v} < p^*$, indeed $\Omega(\underline{v}) > \Pi^*$. Using integration by parts,

$$E[v|v \in [\underline{v}, p^*]] = \frac{\int_{\underline{v}}^{p^*} v f(v)dv}{F(p^*) - F(\underline{v})} = \underline{v} + \frac{\int_{\underline{v}}^{p^*} (F(p^*) - F(v))dv}{F(p^*) - F(\underline{v})}.$$

By the continuity of $f$, the derivative exists for all $\underline{v} \leq p^*$ and equals

$$\frac{d}{d\underline{v}} E[v|v \in [\underline{v}, p^*]] = f(\underline{v}) \frac{\int_{\underline{v}}^{p^*} (F(p^*) - F(v))dv}{[F(p^*) - F(\underline{v})]^2} < f(\underline{v}) \frac{p^* - \underline{v}}{F(p^*) - F(\underline{v})}.$$

Hence, the derivative of $\Omega$ also exists, is continuous, and satisfies

$$\Omega'(\underline{v}) < -f(\underline{v})E[v|v \in [\underline{v}, p^*]] + [1 - F(\underline{v})]f(\underline{v}) \frac{p^* - \underline{v}}{F(p^*) - F(\underline{v})}.$$

Moreover,

$$\Omega'(p^*) = \lim_{\underline{v}\to p^*} \Omega'(\underline{v}) < -f(p^*)p^* + 1 - F(p^*).$$

As $p^* \in \text{argmax}_p[1-F(p)]p$ satisfies the first-order condition $-f(p^*)p^* + 1 - F(p^*) = 0$, we have $\Omega'(p^*) < 0$. Noting that $\Omega(p^*) = \Pi^*$, there thus are $\underline{v} < p^*$ such that $\Omega(\underline{v}) > \Pi^*$.

Fix $\underline{v} < p^*$ such that $\Omega(\underline{v}) > \Pi^*$. Now, either the corresponding information structure is extensionproof and the seller sets price $E[v|v \in [\underline{v}, p^*]] < p^*$ or the seller prefers to extend it. If the seller extends it, she sets a price $p \in (\underline{v}, p^*)$ because all prices $p \notin (\underline{v}, p^*)$ result in a seller payoff of at most $\Pi^*$. In any case, the possibly extended information structure induces a price $p < p^*$, is extensionproof, and perfectly reveals all valuations $v > p^*$. Because of the latter, the CDF of posterior valuations $H$ satisfies $H(v) = F(v)$ for $v \geq p^*$. Therefore, the induced buyer payoff is

$$\int_p^1 (v-p)dH(v) = \int_p^{p^*} (v-p)dH(v) + \int_{p^*}^1 (v-p)dF(v) \geq \int_{p^*}^1 (v-p)dF(v) > \int_{p^*}^1 (v-p^*)dF(v),$$

where the final inequality follows from $p < p^*$. □

**Proof of Lemma 1.** Let $(S^a, (G_v^a))$ be extensionproof. We first extend $(S^a, (G_v^a))$ such that the support of the posterior belief consists of at most two valuations almost surely and the CDF of posterior valuations remains unchanged. The extended information structure, denoted by $(S^{ab}, (G_v^{ab}))$, has signals $(s^a, s^b)$, where $s^b \in S^b = [0,1]^2$. In the following, we define the CDF over $s^b$ conditional on $v$ and $s^a$, assuming without loss of generality that the support of $F_{s^a}$ is not a singleton. Let

$$w(s^a) := E[v|s^a],$$

$$c(s^a) := \int_0^{w(s^a)} (w(s^a) - v)dF_{s^a}(v) = \int_{w(s^a)}^1 (v - w(s^a))dF_{s^a}(v).$$

We write $s^b = (v_L, v_H)$, where $v_L \leq v_H$. If $v \in [0, w(s^a)]$, then $(v_L, v_H)$ is drawn from the set $\{(v_L, v_H) : v_L = v, v_H \in [w(s^a), 1]\}$, where $v_H$ is distributed according to the CDF

$$G^H(v_H|s^a) := \frac{1}{c(s^a)} \int_{w(s^a)}^{v_H} (u_H - w(s^a))dF_{s^a}(u_H). \tag{A.1}$$

If $v \in [w(s^a), 1]$, then $(v_L, v_H)$ is drawn from the set $\{(v_L, v_H) : v_L \in [0, w(s^a)], v_H = v\}$, where $v_L$ is distributed according to the CDF[18]

$$G^L(v_L|s^a) := \frac{1}{c(s^a)} \int_0^{v_L} (w(s^a) - u_L) dF_{s^a}(u_L).$$

Thus, the distribution function of $(v_L, v_H)$ conditional on only $s^a$ (and not $v$) draws signals from $[0, w(s^a)] \times [w(s^a), 1]$ and is given by

$$\begin{aligned}\bar{G}(v_L, v_H|s^a) &= \int_0^{v_L} G^H(v_H|s^a) dF_{s^a}(v) + \int_{w(s^a)}^{v_H} G^L(v_L|s^a) dF_{s^a}(v) \\ &= \frac{1}{c(s^a)} \int_0^{v_L} \int_{w(s^a)}^{v_H} (u_H - u_L) dF_{s^a}(u_H) dF_{s^a}(u_L), \qquad \text{(A.2)}\end{aligned}$$

where the second line follows from Fubini's Theorem. Clearly, under the extended information structure $(S^{ab}, (G_v^{ab}))$ the support of the posterior belief consists of at most two valuations almost surely. Specifically, the posterior belief $F_{s^a,(v_L,v_H)}$ has support $\{v_L, v_H\}$ and is characterized by the probability $F_{s^a,(v_L,v_H)}(v_L)$ that the valuation equals $v_L$. Analogously to the definition of the posterior belief in (1), for all $M_L \in \mathcal{B}([0, w(s^a)])$ and all $M_H \in \mathcal{B}([w(s^a), 1])$ we have

$$\int_{M_L \times M_H} F_{s^a,(v_L,v_H)}(v_L) d\bar{G}(v_L, v_H|s^a) = \int_{M_L} \int_{M_H} dG^H(v_H|s^a) dF_{s^a}(v_L). \qquad \text{(A.3)}$$

Plugging (A.1) and (A.2) into (A.3) gives

$$\begin{aligned}&\frac{1}{c(s^a)} \int_{M_L} \int_{M_H} F_{s^a,(v_L,v_H)}(v_L)(v_H - v_L) dF_{s^a}(v_H) dF_{s^a}(v_L) \\ &\qquad = \frac{1}{c(s^a)} \int_{M_L} \int_{M_H} (v_H - w(s^a)) dF_{s^a}(v_H) dF_{s^a}(v_L).\end{aligned}$$

Since this equation holds for $F_{s^a,(v_L,v_H)}(v_L) = (v_H - w(s^a))/(v_H - v_L)$, and since $F_{s^a,(v_L,v_H)}$ is unique for almost all $(v_L, v_H)$, we have $E[v|s^a, (v_L, v_H)] = w(s^a)$ almost surely.[19] Thus, the extended information structure $(S^{ab}, (G_v^{ab}))$ induces the same CDF of posterior valuations as $(S^a, (G_v^a))$. Consequently, $(S^{ab}, (G_v^{ab}))$ induces the same price, the same

[18] There are other extensions under which the posterior belief also consists of at most two valuations almost surely and the CDF of posterior valuations remains unchanged. The extension presented here is simple in that the CDFs $G^H(\cdot|s^a)$ and $G^L(\cdot|s^a)$ do not depend on $v$.

[19] It suffices to consider only signal sets in the Cartesian product of the $\sigma$-algebras $\mathcal{B}([0, w(s^a)])$ and $\mathcal{B}([w(s^a), 1])$ rather than arbitrary sets in the product $\sigma$-algebra because, for each $v$, one component of the signal $(v_L, v_H)$ is fixed and the other is drawn from $[0, w(s^a)]$ or $[w(s^a), 1]$, respectively.

buyer payoff, and the same seller payoff. Moreover, $(S^{ab}, (G_v^{ab}))$ is also extensionproof because the seller could have performed the extension herself.

Let $p$ be any optimal price for the seller under $(S^{ab}, (G_v^{ab}))$. We now extend $(S^{ab}, (G_v^{ab}))$ to a $p$-pairwise information structure $(S^{abc}, (G_v^{abc}))$. Conditional on signal $(s^a, (v_L, v_H))$, the extension acts as follows:

- If $E[v|s^a, (v_L, v_H)] = p$, then $E[v|s^a, (v_L, v_H), s^c] = p$ (no disclosure).
- If $E[v|s^a, (v_L, v_H)] > p$ and $v_L < p < v_H$, then $E[v|s^a, (v_L, v_H), s^c] \in \{p, v_H\}$ (partial disclosure).
- In all other cases, $E[v|s^a, (v_L, v_H), s^c] \in \{v_L, v_H\}$ (full disclosure).

Clearly, $(S^{abc}, (G_v^{abc}))$ is $p$-pairwise. Note that by the extensionproofness of $(S^{ab}, (G_v^{ab}))$, signals $(s^a, (v_L, v_H))$ with $E[v|s^a, (v_L, v_H)] < p$ and $v_L < p < v_H$ have probability zero. Hence, by construction, $E[v|s^a, (v_L, v_H), s^c] \geq p$ if and only if $E[v|s^a, (v_L, v_H)] \geq p$, and so at price $p$, the buyer payoff and the probability of trade remain unchanged. By the latter, also the seller payoff remains unchanged at $p$. Since $(S^{ab}, (G_v^{ab}))$ is extensionproof, it follows that $p$ remains optimal for the seller under $(S^{abc}, (G_v^{abc}))$ and that $(S^{abc}, (G_v^{abc}))$ is extensionproof as well. □

**Proof of Lemma 2.** In the main text. □

**Proof of Lemma 3.** First, we derive a property of the CDFs $G_v^L$ and $G_v^H$ of $p$-pairwise information structures. Suppose $(S, (G_v))$ is $p$-pairwise. By (4) and (5),

$$\int_0^1 (p - v) dF_s(v) = 0 \quad \text{for almost all } s = (v_L, v_H) \text{ with } v_L < v_H. \tag{A.4}$$

Because a function whose integral is zero on every measurable set is zero almost everywhere, (A.4) holds if and only if

$$\int_{M_L \times M_H} \int_0^1 (p - v) dF_s(v) d\bar{G}(s) = 0 \tag{A.5}$$

for all $M_L \in \mathcal{B}([0,p])$ and all $M_H \in \mathcal{B}([p,1])$.[20] As the support of $F_s$ is contained in $M_L \cup M_H$ for almost all signals $s \in M_L \times M_H$, (A.5) can be written as

$$\begin{aligned}
&\int_{M_L\times M_H}\int_{M_L\cup M_H}(v-p)dF_s(v)d\bar{G}(s)=0\\
\iff\quad &\int_{M_L\times M_H}\int_{M_L}(p-v)dF_s(v)d\bar{G}(s)=\int_{M_L\times M_H}\int_{M_H}(v-p)dF_s(v)d\bar{G}(s)\\
\iff\quad &\int_{M_L}\int_{M_H}dG_v^H(u)(p-v)dF(v)=\int_{M_H}\int_{M_L}dG_v^L(u)(v-p)dF(v) \qquad \text{(A.6)}
\end{aligned}$$

where the last line uses the definition of the posterior belief $F_s$ in (1). Thus, $G_v^L$ and $G_v^H$ belong to a $p$-pairwise information structure if and only if (A.6) for all $M_L$ and $M_H$.

We are now ready to prove the lemma. First, we consider an arbitrary $p$-pairwise information structure $(S,(G_v))$ that is not perfect and show the existence of a unique equivalent $p$-pairwise distribution function $J$. The function $\alpha$ for $J$ is uniquely pinned down by the requirement for equivalence that $\alpha(v)=1-G_v^H(v)$ for $v<p$ and $\alpha(v)=G_v^L(p)$ for $v>p$. Setting $M_L=[0,v_L]$ and $M_H=[p,v_H]$ in (A.6), we obtain

$$\int_0^{v_L}\int_p^{v_H}dG_v^H(u)(p-v)dF(v)=\int_p^{v_H}\int_0^{v_L}dG_v^L(u)(v-p)dF(v). \qquad \text{(A.7)}$$

Hence, the specified $\alpha$ is consistent with the parameter $c$ for $J$ satisfying

$$c=\int_0^p(1-G_v^H(v))(p-v)dF(v)=\int_p^1 G_v^L(p)(v-p)dF(v).$$

Moreover, $c>0$ because $(S,(G_v))$ is not perfect, and thus neither $G_v^L(p)=0$ for almost all $v>p$ nor $1-G_v^H(v)=0$ for almost all $v<p$. Given $c$, the distribution function $J$ is uniquely pinned down by the requirement for equivalence that

$$J(v_L,v_H)=\frac{1}{c}\int_0^{v_L}\int_p^{v_H}dG_v^H(u)(p-v)dF(v). \qquad \text{(A.8)}$$

Using (A.7), the marginals of $J$ are consistent with $\alpha$ and satisfy

$$\begin{aligned}
J(v_L,1)&=\frac{1}{c}\int_0^{v_L}[1-G_v^H(v)](p-v)dF(v),\\
J(p,v_H)&=\frac{1}{c}\int_p^{v_H}G_v^L(p)(v-p)dF(v).
\end{aligned}$$

[20] As in the proof of Lemma 1, it suffices to consider only signal sets in the Cartesian product of the $\sigma$-algebras rather than arbitrary sets in the product $\sigma$-algebra because, for each $v$, $G_v$ is either a distribution function over $[0,p]\times\{v\}$ or over $\{v\}\times[p,1]$.

Hence, the uniquely specified equivalent $J$ is indeed a $p$-pairwise distribution function.

Now, we consider an arbitrary $p$-pairwise distribution function $J$ and construct an equivalent $p$-pairwise information structure $(S,(G_v))$. Let $J^{H|L}_{v_L}$ denote the CDF of $v_H$ over $[p,1]$ conditional on $v_L$, and let $J^{L|H}_{v_H}$ denote the CDF of $v_L$ over $[0,p]$ conditional on $v_H$. $J^{H|L}_{v_L}$ and $J^{L|H}_{v_H}$ are characterized by the condition that

$$\int_{M_L\times M_H} dJ(v_L,v_H) = \int_{M_L}\int_{M_H} dJ^{H|L}_{v_L}(v_H)dJ^L(v_L) = \int_{M_H}\int_{M_L} dJ^{L|H}_{v_H}(v_L)dJ^H(v_H) \quad \text{(A.9)}$$

for all $M_L \in \mathcal{B}([0,p])$ and all $M_H \in \mathcal{B}([p,1])$. As the CDFs $J^{H|L}_{v_L}$ and $J^{L|H}_{v_H}$ correspond to two regular conditional distributions, they exist and are unique almost everywhere (see, e.g., Dudley, 2002, Thm. 10.2.2). We use $J^{H|L}_{v_L}$, $J^{L|H}_{v_H}$, and $\alpha$ of $J$ to define the CDFs $G^H_v$ and $G^L_v$, which suffices to fully specify a $p$-pairwise information structure $(S,(G_v))$. Let

$$\begin{aligned} G^H_v(v_H) &= J^{H|L}_v(v_H)\alpha(v) + 1 - \alpha(v) && \text{for all } v < p, \\ G^L_v(v_L) &= J^{L|H}_v(v_L)\alpha(v) && \text{for all } v > p. \end{aligned}$$

Observe that thus $G^H_v(v) = 1-\alpha(v)$ and $G^L_v(p) = \alpha(v)$. Moreover, using the definition of $J^L$ and setting $M_L = [0,v_L]$ and $M_H = [p,v_H]$, the first equality in (A.9) can be written as (A.8). Consequently, $(S,(G_v))$ is equivalent to $J$. $(S,(G_v))$ is indeed $p$-pairwise because using the definition of $J^L$ and $J^H$, the second equality in (A.9) yields (A.6).

Finally, we show that equivalent $p$-pairwise information structures are unique almost everywhere. Suppose $(S,(G_v))$ and $(S,(\hat{G}_v))$ are $p$-pairwise and equivalent to the same $p$-pairwise distribution function $J$. Then, $1-\alpha(v) = G^H_v(v) = \hat{G}^H_v(v)$ for $v < p$ and $\alpha(v) = G^L_v(p) = \hat{G}^L_v(p)$ for $v > p$. Moreover, for all $v_L \in [0,p]$ and all $v_H \in [p,1]$,

$$\begin{aligned} cJ(v_L,v_H) = {} & \int_0^{v_L}\int_p^{v_H} dG^H_v(u)(p-v)dF(v) = \int_0^{v_L}\int_p^{v_H} d\hat{G}^H_v(u)(p-v)dF(v) && \text{(A.10)} \\ \iff \quad & \int_0^{v_L}[G^H_v(v_H) - G^H_v(v)](p-v)dF(v) = \int_0^{v_L}[\hat{G}^H_v(v_H) - \hat{G}^H_v(v)](p-v)dF(v) \\ \iff \quad & \int_0^{v_L}[G^H_v(v_H) - \hat{G}^H_v(v_H)](p-v)dF(v) = 0. && \text{(A.11)} \end{aligned}$$

As $F$ corresponds to a regular probability measure (see, e.g., Rudin, 1987, Thm. 2.18), every measurable set in $[0,p]$ can be approximated by a countable union of closed balls. Therefore, because $F$ is atomless, (A.11) implies

$$\int_{M_L}[G^H_v(v_H) - \hat{G}^H_v(v_H)](p-v)dF(v) = 0 \quad \text{for every } M_L \in \mathcal{B}([0,p]).$$

Hence, $G_v^H(v_H) = \hat{G}_v^H(v_H)$ for all $v_H \in [p,1]$ and almost all $v \in [0,p]$. On the other hand, as $(S,(G_v))$ and $(S,(\hat{G}_v))$ both satisfy (A.6), (A.10) also implies

$$\int_p^{v_H}\int_0^{v_L} dG_v^L(u)(v-p)dF(v) = \int_p^{v_H}\int_0^{v_L} d\hat{G}_v^L(u)(v-p)dF(v).$$

Proceeding from this as we did for $G_v^H$ and $\hat{G}_v^H$, we find that $G_v^L(v_L) = \hat{G}_v^L(v_L)$ for all $v_L \in [0,p]$ and almost all $v \in [p,1]$. Consequently, every $p$-pairwise distribution function is equivalent to an almost everywhere unique $p$-pairwise information structure. □

**Proof of Lemma 4.** Let $J$ be an extensionproof $p$-pairwise distribution function that induces price $p$. Denote by $C$ the copula of $J$, that is, $J(v_L,v_H) = C(J^L(v_L),J^H(v_H))$. For $\underline{v}$ such that

$$\int_{\underline{v}}^p (p-v)dF(v) = \int_0^p \alpha(v)(p-v)dF(v) = c, \tag{A.12}$$

let $\widetilde{J}$ be a $(p,\underline{v})$-pairwise distribution function that also has copula $C$, which exists by Sklar's Theorem. The buyer payoff (6) at price $p$ is $\int_p^1(v-p)dF(v) - c$ under both $\widetilde{J}$ and $J$. The seller payoff (7) at price $p$ is weakly higher under $\widetilde{J}$ than under $J$ because

$$\int_{\underline{v}}^p dF(v) \geq \int_0^p \alpha(v)dF(v). \tag{A.13}$$

To see this, suppose, by contradiction, that (A.13) does not hold. Then (A.12) implies

$$\frac{1}{\int_{\underline{v}}^p dF(v)}\int_{\underline{v}}^p (p-v)dF(v) > \frac{1}{\int_0^p \alpha(v)dF(v)}\int_0^p \alpha(v)(p-v)dF(v).$$

This is a contradiction as $p-v$ is decreasing in $v$ and the CDF $F(v)/[\int_{\underline{v}}^p dF(u)]$ over $[\underline{v},p]$ first-order stochastically dominates the CDF $[\int_0^v \alpha(u)dF(u)]/[\int_0^p \alpha(u)dF(u)]$ over $[0,p]$.

Since $J$ is extensionproof and induces price $p$, it induces a seller payoff that is at least as high as the perfect-information payoff $\Pi^* = \max_q[1-F(q)]q$. Since under $\widetilde{J}$, the seller payoff at any price $q \notin [\underline{v},\overline{v}]$ equals $[1-F(q)]q$, it follows that $\widetilde{J}$ induces price $p$. It remains to show that $\widetilde{J}$ is extensionproof.

Observe that for $v_L \in [0,\underline{v}]$, we have $J^L(v_L) \geq 0 = \widetilde{J}^L(v_L)$, whereas for $v_L \in [\underline{v},p]$,

$$1 - J^L(v_L) = \frac{1}{c}\int_{v_L}^p \alpha(v)(p-v)dF(v) \leq \frac{1}{c}\int_{v_L}^p (p-v)dF(v) = 1 - \widetilde{J}^L(v_L).$$

Moreover, for $v_H \in [p,\overline{v}]$,

$$J^H(v_H) = \frac{1}{c}\int_p^{v_H} \alpha(v)(v-p)dF(v) \leq \frac{1}{c}\int_p^{v_H} (v-p)dF(v) = \widetilde{J}^H(v_H),$$

whereas for $v_H \in [\overline{v}, 1]$, $J^H(v_H) \leq 1 = \widetilde{J}^H(v_H)$. Consequently,

$$J^L(v_L) \geq \widetilde{J}^L(v_L) \text{ for all } v_L \qquad \text{and} \qquad J^H(v_H) \leq \widetilde{J}^H(v_H) \text{ for all } v_H. \tag{A.14}$$

Consider an arbitrary $p$-pairwise distribution function $\widehat{J}$. Using (8), the gain in seller payoff when the seller charges price $q > p$ and performs the $q$-optimal extension instead of charging price $p$ and not performing any extension equals

$$\begin{aligned}\left[\int_S \phi_q(v_L, v_H) d\widehat{J}(v_L, v_H) + 1 - F(q)\right] q - \left[\int_S \phi_p(v_L, v_H) d\widehat{J}(v_L, v_H) + 1 - F(p)\right] p \\ = \int_S [\phi_q(v_L, v_H) q - \phi_p(v_L, v_H) p] d\widehat{J}(v_L, v_H) + [1 - F(q)]q - [1 - F(p)]p. \end{aligned} \tag{A.15}$$

We will show that the integral in (A.15) is smaller for $\widehat{J} = \widetilde{J}$ than for $\widehat{J} = J$. Accordingly, because $J$ is extensionproof, also $\widetilde{J}$ is extensionproof. We will use the shorthand notation

$$\delta(v_L, v_H) := \phi_q(v_L, v_H) q - \phi_p(v_L, v_H) p = \max\left\{\frac{q(v_H - q)c}{(v_H - p)(q - v_L)}, 0\right\} - \frac{pc}{(p - v_L)}.$$

The function $\delta$ is decreasing in $v_L$ and increasing in $v_H$. This can be seen as follows. For $v_H \leq q$, $\partial\delta(v_L, v_H)/\partial v_L = -pc(p - v_L)^{-2} < 0$. For $v_H > q$,

$$\begin{aligned}\frac{\partial}{\partial v_L}\delta(v_L, v_H) &= \frac{q(v_H - q)c}{(v_H - p)(q - v_L)^2} - \frac{pc}{(p - v_L)^2} \\ &= \frac{(v_H - q)q(p - v_L)^2 - (v_H - p)p(q - v_L)^2}{(v_H - p)(q - v_L)^2(p - v_L)^2} c < 0,\end{aligned}$$

where the numerator is negative because $q > p$ implies $(v_H - q) < (v_H - p)$, $q(p - v_L) < p(q - v_L)$, and $(p - v_L) < (q - v_L)$. Moreover, $\partial\delta(v_L, v_H)/\partial v_H = \partial\phi_q(v_L, v_H)q/\partial v_H \geq 0$.

Define $\overline{v}_L := p - v_L$. Let $K$ be the joint distribution function of $\overline{v}_L$ and $v_H$ that is implied by $J$. The marginals of $K$ are $K^L(\overline{v}_L) = 1 - J^L(p - \overline{v}_L)$ and $K^H(v_H) = J^H(v_H)$. Let $D$ be the copula of $K$ and recall that $C$ is the copula of $J$. By Nelsen (2006, Thm. 2.4.4), $D(u_1, u_2) = u_2 - C(1 - u_1, u_2)$. Let $\widetilde{K}$, $\widetilde{K}^L$, and $\widetilde{K}^H$ be the corresponding distribution functions implied by $\widetilde{J}$. Note that $\widetilde{K}$ also has copula $D$.

Because of (A.14), we have $K^L(\overline{v}_L) \leq \widetilde{K}^L(\overline{v}_L)$ for all $\overline{v}_L$ and $K^H(v_H) \leq \widetilde{K}^H(v_H)$ for all $v_H$. Together with the fact that $K$ and $\widetilde{K}$ have a common copula, this implies according to Shaked and Shanthikumar (2007, Thm. 6.B.14) that $\widetilde{K}$ is smaller than $K$

in the usual stochastic order. Hence, since $\delta(p-\overline{v}_L, v_H)$ is increasing in $\overline{v}_L$ and $v_H$,

$$\int_S \delta(v_L, v_H) d\widetilde{J}(v_L, v_H) = \int_S \delta(p-\overline{v}_L, v_H) d\widetilde{K}(\overline{v}_L, v_H)$$
$$\leq \int_S \delta(p-\overline{v}_L, v_H) dK(\overline{v}_L, v_H) = \int_S \delta(v_L, v_H) dJ(v_L, v_H).$$

Consequently, the integral in (A.15) is smaller for $\widehat{J} = \widetilde{J}$ than for $\widehat{J} = J$. □

**Proof of Theorem 1.** In the main text. □

**Proof of Proposition 2.** We first prove that optimal $(p, \underline{v})$ exist. The objective function $\int_{\underline{v}}^{1} (v-p) dF(v)$ in problem (13) is continuous in $(p, \underline{v})$. The set of $(p, \underline{v})$ that satisfy (11) and (12) is nonempty because it contains $(p, \underline{v}) = (p^*, p^*)$, where $p^*$ is the lowest price that is optimal for the seller under perfect information (as defined in Section 3). We now show that the set of $(p, \underline{v})$ that satisfy (11) and (12) is closed. Let $((p^k, \underline{v}^k))$ be a sequence such that $(p^k, \underline{v}^k)$ satisfies (11) and (12) for all $k$ and $(p^k, \underline{v}^k) \to (p', \underline{v}')$. $(p', \underline{v}')$ satisfies (11) because the constraint function in (11), $[1-F(\underline{v})]p - \Pi^*$, is continuous in $(p, \underline{v})$. Note that since $\int_{\underline{v}}^{l} (p-v) dF(v)$ is continuous in $(l, p, \underline{v})$ and strictly increasing in $l$, $\mu_p(\underline{v})$ is continuous in $(p, \underline{v})$. For every $q \in (p', \mu_{p'}(\underline{v}'))$, there must then exist $k'$ such that $q \in (p^k, \mu_{p^k}(\underline{v}^k))$ for all $k > k'$. Now, the constraint function in (12),

$$[1-F(\underline{v})]p - \left[1 - F(q) + \int_{\underline{v}}^{\mu_p(q)} x_q(v, \mu_p(v)) dF(v)\right] q,$$

is continuous in $(p, \underline{v})$ by the Dominated Convergence Theorem since

$$x_q(v, \mu_p(v)) f(v) \mathbf{1}_{v \in [\underline{v}, \mu_p(q)]} \leq f(v),$$

where $\mathbf{1}$ denotes the indicator function. Hence, $(p', \underline{v}')$ also satisfies (12). Consequently, problem (13) has a solution as it corresponds to maximizing a continuous function over a nonempty, closed, and bounded set.

Case (i) is proved in the main text directly following the proposition. To prove case (ii), we first establish two claims. The first claim says that for a $(p, \underline{v})$-negative-assortative information structure with fixed $p$, the seller's gain from setting price $q$ and performing the $q$-optimal extension is strictly increasing in $\underline{v}$.

**Claim A1.** *Let* $\underline{v}' < \underline{v} \leq p$ *and* $q \in (p, \mu_p(\underline{v})]$. *Then,*

$$\Psi(q, p, \underline{v}) - [1 - F(\underline{v})]p > \Psi(q, p, \underline{v}') - [1 - F(\underline{v}')]p.$$

*Proof.* We have

$$\Psi(q, p, \underline{v}) - \Psi(q, p, \underline{v}') + [F(\underline{v}) - F(\underline{v}')]\, p = \int_{\underline{v}'}^{\underline{v}} p - q x_q(v, \mu_p(v)) dF(v) > 0,$$

where the inequality follows from

$$x_q(v, \mu_p(v)) = \frac{(p - v)[\mu_p(v) - q]}{(q - v)[\mu_p(v) - p]} < \frac{p - v}{q - v} \leq \frac{p}{q}. \qquad \square$$

The second claim says that decreasing $\underline{v}$ strictly relaxes (11) and (12).

**Claim A2.** *Suppose* $(p, \underline{v})$ *satisfies* (11) *and* (12). *Then for every* $\underline{v}' < \underline{v}$,

$$[1 - F(\underline{v}')]p > \Pi^*, \tag{A.16}$$

$$[1 - F(\underline{v}')]p > \Psi(q, p, \underline{v}') \quad \textit{for all } q \in (p, \mu_p(\underline{v}')). \tag{A.17}$$

*Proof.* (A.16) holds because $(p, \underline{v})$ satisfies (11) and $1 - F(\underline{v}') > 1 - F(\underline{v})$. Since $(p, \underline{v})$ satisfies (12), Claim A1 implies

$$\Psi(q, p, \underline{v}') - [1 - F(\underline{v}')]p < 0 \quad \text{for all } q \in (p, \mu_p(\underline{v})]. \tag{A.18}$$

By contradiction, suppose there exist $q \in (\mu_p(\underline{v}), \mu_p(\underline{v}'))$ such that

$$\Psi(q, p, \underline{v}') - [1 - F(\underline{v}')]p \geq 0.$$

Since $q < \mu_p(\underline{v}')$ is equivalent to $\mu_p(q) > \underline{v}'$, Claim A1 implies

$$\Psi(q, p, \mu_p(q)) - [1 - F(\mu_p(q))]p > 0.$$

This results in a contradiction because

$$\Psi(q, p, \mu_p(q)) = [1 - F(\mu_p(q))]\mu_p(q) \leq \Pi^*,$$

whereas

$$[1 - F(\mu_p(q))]p > [1 - F(\underline{v})]p \geq \Pi^*,$$

where the first inequality follows from the fact that $q > \mu_p(\underline{v})$ implies $\mu_p(q) < \underline{v}$ and the second inequality follows from the assumption that $(p, \underline{v})$ satisfies (11). Hence,

$$\Psi(q, p, \underline{v}') - [1 - F(\underline{v}')]p < 0 \quad \text{for all } q \in (\mu_p(\underline{v}), \mu_p(\underline{v}')). \tag{A.19}$$

Taken together, (A.18) and (A.19) give (A.17). □

Now, consider case (ii). Recall from the first part of the proof that the constraint functions in (11) and (12) are continuous in $(p, \underline{v})$. Hence, the sets used in the definition of $\omega$ and of $\hat{p}(\underline{v})$ are closed and bounded. Accordingly, $\omega$ and $\hat{p}(\underline{v})$ exist if these sets are nonempty. Indeed, $\omega$ exists because $(p, \underline{v}) = (\Pi^*/[1 - F(p^*)], p^*) = (p^*, p^*)$ satisfies (12). Moreover, $\hat{p}(\omega) = \Pi^*/[1 - F(\omega)]$, and $\hat{p}(\underline{v})$ exists for $\underline{v} < \omega$ because $(\hat{p}(\omega), \underline{v})$ satisfies (11) and (12) by Claim A2. Because $(p, \underline{v}) = (\Pi^*, 0)$ violates (12), $\omega > 0$ and $\hat{p}(0) > \Pi^*$.

Next, we prove that $\hat{p}(\underline{v})$ is strictly increasing. By contradiction, suppose there exist $\underline{v}', \underline{v}'' \in [0, \omega]$ with $\underline{v}' < \underline{v}''$ and $\hat{p}(\underline{v}') \geq \hat{p}(\underline{v}'')$. By Claim A2, $(\underline{v}', \hat{p}(\underline{v}''))$ satisfies (11) and (12) with strict inequality, which contradicts the definition of $\hat{p}(\underline{v}')$ as the lowest feasible price given $\underline{v}'$.

Finally, suppose $(p, \underline{v})$ is optimal. Note that any $(p', \underline{v}')$ that satisfies (11) and (12) implements social surplus $\int_{\underline{v}'}^1 v dF(v)$ and a seller payoff of at least $\Pi^*$. If $\underline{v}' > \omega$, the implemented buyer payoff is thus at most $\int_{\underline{v}'}^1 v dF(v) - \Pi^* < \int_{\omega}^1 v dF(v) - \Pi^*$, where the right-hand side is the buyer payoff implemented by $(\hat{p}(\omega), \omega)$. Hence, $\underline{v} \in [0, \omega]$. Because the buyer payoff $\int_{\underline{v}}^1 (v - p') dF(v)$ is strictly decreasing in $p'$, we have $p = \hat{p}(\underline{v})$. As $(p, \underline{v}) \neq (\Pi^*, 0)$, the implemented buyer payoff is strictly less than $\overline{U}$. □

**Proof of Lemma 5.** Recall that under a regular prior, the PDF $f$ is twice differentiable and satisfies (14). As $(p, \underline{v})$ remains fixed, in what follows we write $\Psi(q)$ instead of $\Psi(q, p, \underline{v})$ and $\mu(q)$ instead of $\mu_p(q)$. Moreover, let

$$A(q) := \int_{\underline{v}}^{\mu(q)} x_q(v, \mu(v)) dF(v) = \int_{\underline{v}}^{\mu(q)} \frac{p - v}{\mu(v) - p} \frac{\mu(v) - q}{q - v} dF(v).$$

Then, $\Psi(q) = [1 - F(q)]q + A(q)q$ and

$$\Psi'''(q) = A'''(q)q + 3A''(q) - f''(q)q - 3f'(q). \tag{A.20}$$

First, we show that $\Psi'''(q) \leq 0$ for $q \in (p, \mu(\underline{v}))$. Observe that by (11), we have $p \geq \Pi^*$; hence, to all $q \in (p, \mu(\underline{v}))$ the inequality in (14) applies.

Noting that the integrand in $A(q)$ vanishes at $v = \mu(q)$, we have

$$A'(q) = -\int_{\underline{v}}^{\mu(q)} \frac{p-v}{\mu(v)-p} \frac{\mu(v)-v}{(q-v)^2} dF(v). \tag{A.21}$$

The second derivative is

$$\begin{aligned} A''(q) &= 2\int_{\underline{v}}^{\mu(q)} \frac{p-v}{\mu(v)-p} \frac{\mu(v)-v}{(q-v)^3} dF(v) - \frac{p-\mu(q)}{q-p} \frac{f(\mu(q))}{q-\mu(q)} \mu'(q) \\ &= 2\int_{\underline{v}}^{\mu(q)} \frac{p-v}{\mu(v)-p} \frac{\mu(v)-v}{(q-v)^3} dF(v) + \frac{f(q)}{q-\mu(q)}, \end{aligned}$$

where the second line uses

$$(p-\mu(v))f(\mu(v))\mu'(v) = (p-v)f(v), \tag{A.22}$$

which follows from the definition of $\mu$ in (9) when taking the derivative with respect to $v$. Using (A.22) once more,

$$A'''(q) = -6\int_{\underline{v}}^{\mu(q)} \frac{p-v}{\mu(v)-p} \frac{\mu(v)-v}{(q-v)^4} dF(v) - \frac{3-\mu'(q)}{(q-\mu(q))^2} f(q) + \frac{f'(q)}{q-\mu(q)}.$$

So by (A.20),

$$\begin{aligned} \Psi'''(q) = &-6\int_{\underline{v}}^{\mu(q)} \frac{p-v}{\mu(v)-p} \frac{\mu(v)-v}{(q-v)^4} v dF(v) + \frac{\mu'(q)q}{(q-\mu(q))^2} f(q) \\ &- \frac{3\mu(q)}{(q-\mu(q))^2} f(q) + \frac{\mu(q)}{q-\mu(q)} f'(q) - 2f'(q) - f''(q)q. \end{aligned} \tag{A.23}$$

The first line of (A.23) is clearly nonpositive. We will show that the second line is nonpositive as well, and thus $\Psi'''(q) \leq 0$.

First, suppose $f'(q) \leq 0$. Then (14) is equivalent to $f''(q)q \geq -2f'(q)$, which guarantees that the second line of (A.23) is indeed nonpositive.

Now, suppose $f'(q) > 0$. The second line of (A.23) is nonpositive if and only if

$$3\mu(q)f(q) - \mu(q)(q-\mu(q))f'(q) + (q-\mu(q))^2[2f'(q) + f''(q)q] \geq 0.$$

Define $\gamma := \frac{\mu(q)}{q}$ and divide by $q$ to get

$$R(\gamma) := 3\gamma f(q) - \gamma(1-\gamma)f'(q)q + (1-\gamma)^2[2f'(q)q + f''(q)q^2] \geq 0.$$

Since $\mu(p) = p$, $\mu(\mu(\underline{v})) = \underline{v}$, and $\mu$ is decreasing, $\gamma \in [0,1]$ for all $q \in [p, \mu(\underline{v})]$. We will show that $R(\gamma) \geq 0$ for all $\gamma \in [0,1]$, which implies that the second line of (A.23) is nonpositive.

$R$ is a quadratic function of the form $R(\gamma) = a_0 + a_1\gamma + a_2\gamma^2$ with coefficients

$$a_0 := 2f'(q)q + f''(q)q^2, \quad a_1 := 3f(q) - f'(q)q - 2a_0, \quad a_2 := f'(q)q + a_0.$$

By (14), $a_0 \geq 0$ and $a_2 > 0$. Hence, $R$ is strictly convex. Moreover, $R(0) \geq 0$ and $R(1) > 0$. Observe that $R'(0) = a_1$. If $a_1 \geq 0$, $R(\gamma) \geq R(0) \geq 0$ for all $\gamma$. If $a_1 < 0$, $R(\gamma) \geq 0$ for all $\gamma$ if and only if $R$ does not have two real roots, that is, if and only if the discriminant $a_1^2 - 4a_0a_2$ is nonpositive. Taken together, we are left to show that $a_1 \geq -2\sqrt{a_0a_2}$ or, equivalently,

$$3f(q) - f'(q)q \geq -2\Big(\sqrt{a_0(a_0 + f'(q)q)} - a_0\Big). \tag{A.24}$$

If $3f(q) - f'(q)q \geq 0$, (A.24) holds because the right-hand side is nonpositive. If $3f(q) - f'(q)q < 0$, then (14) implies

$$a_0 \geq \frac{f'(q)q(f'(q)q - 3f(q))}{12f(q)} > \frac{(f'(q)q - 3f(q))^2}{12f(q)}.$$

It is straightforward to verify that (A.24) holds with equality for $a_0 = \frac{(f'(q)q-3f(q))^2}{12f(q)}$. Moreover, since the right-hand side of (A.24) is decreasing in $a_0$,[21] $a_0 \geq \frac{f'(q)q(f'(q)q-3f(q))}{12f(q)}$ is sufficient for (A.24). This completes the proof that $\Psi'''(q) \leq 0$.

Now, we show the properties of the first derivative, that is, of

$$\Psi'(q) = 1 - F(q) - f(q)q + A'(q)q + A(q). \tag{A.25}$$

Note that $1 - F(p) - f(p)p$ is bounded and $A(p) \leq F(p)$. From (A.21),

$$A'(q) = -\int_{\underline{v}}^{\mu(q)} \left(\frac{p-v}{(q-v)^2} + \frac{(p-v)^2}{(\mu(v)-p)(q-v)^2}\right) dF(v).$$

Consequently, along every decreasing sequence of values $q$ converging to $p$,

$$\begin{aligned}
\lim_{q\downarrow p} A'(q) &\leq \lim_{q\downarrow p} -\int_{\underline{v}}^{\mu(q)} \frac{p-v}{(q-v)^2} f(v)dv \\
&= -\int_{\underline{v}}^{p} \frac{1}{p-v} f(v)dv \\
&\leq -\min_{z\in[\underline{v},p]} f(z) \int_{\underline{v}}^{p} \frac{1}{p-v}dv = -\infty,
\end{aligned}$$

[21]By the inequality of arithmetic and geometric means, $\frac{\partial}{\partial a_0}\sqrt{a_0(a_0 + f'(q)q)} = \frac{\frac{1}{2}(a_0+a_0+f'(q)q)}{\sqrt{a_0(a_0+f'(q)q)}} \geq 1$.

where the first equality follows from $\mu(p) = p$ and from the Monotone Convergence Theorem as $(p-v)(q-v)^{-2}\mathbf{1}_{v\in[\underline{v},\mu(q)]}$ is decreasing in $q$, and where the last equality follows from $1/(p-v) = -d\ln(p-v)/dv$. Hence, (A.25) implies that $\lim_{q\downarrow p}\Psi'(q) = -\infty$.

Finally, since $A(\mu(\underline{v})) = A'(\mu(\underline{v})) = 0$, (A.25) implies that $\Psi'(\mu(\underline{v}))$ equals the derivative of $[1-F(q)]q$ at $q = \mu(\underline{v})$. □

**Proof of Proposition 3.** Suppose $F$ is regular. Then, as we have shown in the main text, $(p,\underline{v}) = (\Pi^*, 0)$ satisfies (12) if and only if $E[v|v \le p^*] \ge \Pi^*$. Hence, case (i) immediately follows from case (i) of Proposition 2.

Consider case (ii) where $E[v|v \le p^*] < \Pi^*$. Hence, case (ii) of Proposition 2 applies, and optimal $(p,\underline{v})$ implement a buyer payoff strictly less than $\overline{U}$. To prove the additional results for this case, we will need the following claim. It says that under the $(\hat{p}(\omega),\omega)$-negative-assortative information structure, the pooling interval is $(\omega, p^*)$ and the seller's extension payoff has no interior maximum.

**Claim A3.** *Suppose the prior $F$ is regular. Then, $\mu_{\hat{p}(\omega)}(\omega) = p^*$ and*

$$[1-F(\omega)]\hat{p}(\omega) = \Psi(\mu_{\hat{p}(\omega)}(\omega), \hat{p}(\omega), \omega) > \Psi(q, \hat{p}(\omega), \omega) \quad \textit{for all } q \in (\hat{p}(\omega), \mu_{\hat{p}(\omega)}(\omega)).$$

*Proof.* First, we show that $\mu_{\hat{p}(\omega)}(\omega) = p^*$. By contradiction, suppose $\mu_{\hat{p}(\omega)}(\omega) \ne p^*$. If $\mu_{\hat{p}(\omega)}(\omega) > p^*$, then

$$\Psi(p^*, \hat{p}(\omega), \omega) > \Pi^* = [1-F(\omega)]\hat{p}(\omega),$$

where the equality follows from the definition of $\omega$ and $\hat{p}$. Hence, $(\hat{p}(\omega),\omega)$ violates the extensionproofness constraint (12), which is a contradiction. So let $\mu_{\hat{p}(\omega)}(\omega) < p^*$. Because $\mu_p(\underline{v})$ is continuous in $(p,\underline{v})$ (see the proof of Proposition 2), there exist $(p',\underline{v}')$ where $\underline{v}' < \omega$ and $p' = \Pi^*/[1-F(\underline{v}')]$ such that $\mu_{p'}(\underline{v}') < p^*$. Note that $(p',\underline{v}')$ satisfies (11). Then, as shown in the main text (directly after Lemma 5), $\mu_{p'}(\underline{v}') < p^*$ implies that $(p',\underline{v}')$ also satisfies (12). This is a contradiction because $\omega$ is the lowest $\underline{v}$ such that $(\Pi^*/[1-F(\underline{v})],\underline{v})$ satisfies (12).

Now, since $\mu_{\hat{p}(\omega)}(\omega) = p^*$, $\Psi(\mu_{\hat{p}(\omega)}(\omega), \hat{p}(\omega), \omega) = \Pi^* = [1-F(\omega)]\hat{p}(\omega)$. The inequality in Claim A3 holds because $(\hat{p}(\omega),\omega)$ satisfies (12) and because by Lemma 5, $\Psi(\cdot, \hat{p}(\omega), \omega)$ is strictly decreasing for $q$ close to $\hat{p}(\omega)$ and $\partial^3\Psi(q, \hat{p}(\omega), \omega)/\partial q^3 \le 0$ for all $q$. □

As $\mu_{\hat{p}(\omega)}(\omega) = p^*$ by Claim A3, we have $\hat{p}(\omega) = E[v|\omega \leq v \leq p^*]$ and therefore $[1-F(\omega)]E[v|\omega \leq v \leq p^*] = \Pi^*$. In fact, any $(p, \underline{v})$ such that $p = E[v|\underline{v} \leq v \leq p^*]$ and $[1-F(\underline{v})]E[v|\underline{v} \leq v \leq p^*] = \Pi^*$ satisfies (12) because of Lemma 5. Hence, $\omega$ must be the smallest $\underline{v}$ with these properties, that is,

$$\omega = \min\{\, \underline{v} : [1-F(\underline{v})]E[v|\underline{v} \leq v \leq p^*] = \Pi^* \,\},$$

as stated in the proposition. Assuming the PDF $f$ to be continuous, as is true for regular $F$, we have shown in the proof of Proposition 1 (see page 35) for the function $\Omega(\underline{v}) = [1-F(\underline{v})]E[v|\underline{v} \leq v \leq p^*]$ that there are $\underline{v}' < p^*$ such that $\Omega(\underline{v}') > \Pi^*$. Now, since $\Omega(0) = E[v|v \leq p^*] < \Pi^*$ and $\Omega$ is continuous, we have $\omega < p^*$.

In the remainder of the proof we show that $(p, \underline{v}) = (\hat{p}(\omega), \omega)$ is not optimal. By Proposition 2, optimal $(p, \underline{v})$ therefore implement a seller payoff strictly greater than $\Pi^*$. Moreover, the implemented buyer payoff under optimal $(p, \underline{v})$ is thus strictly greater than under $(\hat{p}(\omega), \omega)$, that is, strictly greater than $\int_\omega^1 v dF(v) - \Pi^* > \int_{p^*}^1 (v - p^*) dF(v)$, where the inequality follows from $\omega < p^*$.

Since the function $\hat{p}$ is increasing, it is differentiable at almost all $\underline{v} \in (0, \omega)$. The following claim provides a limit of the derivative of $\hat{p}$ as $\underline{v}$ approaches $\omega$.

**Claim A4.** *Suppose the prior $F$ is regular. Then,*

$$\lim_{\underline{v} \to \omega} \frac{d\hat{p}(\underline{v})}{d\underline{v}} = \frac{f(\omega)\hat{p}(\omega)}{1-F(\omega)}. \tag{A.26}$$

*Proof.* Consider $\underline{v} \in (0, \omega)$, and recall that in this case $(\hat{p}(\underline{v}), \underline{v})$ implements seller payoff $[1-F(\underline{v})]\hat{p}(\underline{v}) > \Pi^*$. As $\hat{p}(\underline{v})$ is by definition the lowest feasible price given $\underline{v}$ and as (11) is not binding, (12) must be binding (for otherwise the price could be reduced): there must exist $q \in (\hat{p}(\underline{v}), \mu_{\hat{p}(\underline{v}}(\underline{v}))$ such that $\Psi(q, \hat{p}(\underline{v}), \underline{v}) = [1-F(\underline{v})]\hat{p}(\underline{v})$ and such that $\Psi(\cdot, \hat{p}(\underline{v}), \underline{v})$ is maximized at $q$. Indeed, this (interior) maximizer is unique because $\Psi(\cdot, p, \underline{v})$ is strictly decreasing at $q$ close to $p$ and $\partial^3\Psi(q, p, \underline{v})/\partial q^3 \leq 0$ by Lemma 5. We denote this unique maximizer by $\hat{q}(\underline{v})$. Consequently, for any $\underline{v} \in (0, \omega)$, we have

$$\Psi(\hat{q}(\underline{v}), \hat{p}(\underline{v}), \underline{v}) = [1-F(\underline{v})]\hat{p}(\underline{v}) \quad \text{and} \quad \left.\frac{\partial \Psi(q, \hat{p}(\underline{v}), \underline{v})}{\partial q}\right|_{q=\hat{q}(\underline{v})} = 0.$$

Our goal is to obtain the derivative $d\hat{p}(\underline{v})/d\underline{v}$ using the Implicit Function Theorem.

For $\underline{v} \in (0,\omega)$, $p \in (\underline{v}, E[v|v \in [\underline{v},1])$, and $q \in (p, \mu_p(\underline{v}))$, define the function

$$\Phi(q,p,\underline{v}) := \begin{pmatrix} [1-F(\underline{v})]p - \Psi(q,p,\underline{v}) \\ \frac{\partial \Psi(q,p,\underline{v})}{\partial q} \end{pmatrix}.$$

Observe that for every $\underline{v} \in (0,\omega)$, $(q,p) = (\hat{q}(\underline{v}), \hat{p}(\underline{v}))$ solves $\Phi(q,p,\underline{v}) = 0$. To prepare for applying the Implicit Function Theorem to the latter equation, we first determine properties of the function $\Phi$. The derivative of $\Phi$ with respect to $(q,p)$ is given by

$$D_{q,p}\Phi(q,p,\underline{v}) = \begin{pmatrix} -\frac{\partial \Psi(q,p,\underline{v})}{\partial q} & 1-F(\underline{v}) - \frac{\partial \Psi(q,p,\underline{v})}{\partial p} \\ \frac{\partial^2 \Psi(q,p,\underline{v})}{\partial q^2} & \frac{\partial^2 \Psi(q,p,\underline{v})}{\partial q \partial p} \end{pmatrix}, \tag{A.27}$$

and the derivative with respect to $\underline{v}$ is given by

$$D_{\underline{v}}\Phi(q,p,\underline{v}) = \begin{pmatrix} -f(\underline{v})p - \frac{\partial \Psi(q,p,\underline{v})}{\partial \underline{v}} \\ \frac{\partial^2 \Psi(q,p,\underline{v})}{\partial q \partial \underline{v}} \end{pmatrix}. \tag{A.28}$$

Under a regular prior, the PDF $f$ is twice differentiable. As may be checked, this implies that all partial derivatives in (A.27) and (A.28) exist and are continuous in $(q,p,\underline{v})$. That is, $\Phi$ is continuously differentiable.

To apply the Implicit Function Theorem, we next show that for $\underline{v}$ close to $\omega$, the determinant $|D_{q,p}\Phi(q,p,\underline{v})|$ is not zero at $(q,p) = (\hat{q}(\underline{v}), \hat{p}(\underline{v}))$. Because $\partial^2\Psi(q,p,\underline{v})/\partial q^2 < 0$ and because $\partial\Psi(q,p,\underline{v})/\partial q = 0$ at $(q,p) = (\hat{q}(\underline{v}), \hat{p}(\underline{v}))$, it suffices to show that

$$\lim_{\underline{v}\to\omega} \left. \frac{\partial \Psi(q,p,\underline{v})}{\partial p} \right|_{(q,p)=(\hat{q}(\underline{v}),\hat{p}(\underline{v}))} = 0. \tag{A.29}$$

As the integrand in $\Psi(q,p,\underline{v})$ vanishes at $v = \mu_p(q)$, we have

$$\frac{\partial \Psi(q,p,\underline{v})}{\partial p} = \int_{\underline{v}}^{\mu_p(q)} \frac{\partial x_q(v,\mu_p(v))}{\partial p} + \frac{\partial x_q(v,\mu_p(v))}{\partial \mu_p(v)} \frac{\partial \mu_p(v)}{\partial p} dF(v),$$

where

$$\frac{\partial x_q(v,\mu_p(v))}{\partial p} = \frac{\mu_p(v)-v}{[\mu_p(v)-p]^2} \frac{\mu_p(v)-q}{q-v},$$

$$\frac{\partial x_q(v,\mu_p(v))}{\partial \mu_p(v)} \frac{\partial \mu_p(v)}{\partial p} = \frac{q-p}{[\mu_p(v)-p]^2} \frac{p-v}{q-v} \frac{F(\mu_p(v))-F(v)}{[\mu_p(v)-p]f(\mu_p(v))}.$$

To prove (A.29), we first show that $\hat{p}(\underline{v})$ converges to $\hat{p}(\omega)$ and $\hat{q}(\underline{v})$ to $\mu_{\hat{p}(\omega)}(\omega)$. Indeed, $\lim_{\underline{v}\to\omega} \hat{p}(\underline{v})$ exists and equals $\hat{p}(\omega)$ because if $\liminf_{\underline{v}\to\omega} \hat{p}(\underline{v}) < \hat{p}(\omega)$, then eventually $\hat{p}(\underline{v}) < \Pi^*/[1-F(\underline{v})]$, contradicting that $\hat{p}(\underline{v})$ satisfies (11). To see that $\lim_{\underline{v}\to\omega} \hat{q}(\underline{v})$

exists and equals $\mu_{\hat{p}(\omega)}(\omega)$, note that the function $\Psi(\mu_{\hat{p}(\cdot)}(\cdot), \hat{p}(\cdot), \cdot) - [1 - F(\cdot)]\hat{p}(\cdot)$ is continuous at $\omega$, and so $\lim_{\underline{v}\to\omega} \Psi(\mu_{\hat{p}(\underline{v})}(\underline{v}), \hat{p}(\underline{v}), \underline{v})) = [1 - F(\omega)]\hat{p}(\omega)$ by Claim A3. If $\liminf_{\underline{v}\to\omega} \hat{q}(\underline{v}) < \mu_{\hat{p}(\omega)}(\omega)$, it follows that, along a subsequence, $\Psi(\cdot, \hat{p}(\underline{v}), \underline{v})$ could eventually not have a negative third derivative, contradicting Lemma 5.

Now, recall $\omega < p^*$, which implies $\hat{p}(\omega) > \omega$ and therefore $\mu_{\hat{p}(\omega)}(\omega) > \hat{p}(\omega)$. As $\lim_{\underline{v}\to\omega} \hat{p}(\underline{v}) = \hat{p}(\omega)$ and $\lim_{\underline{v}\to\omega} \hat{q}(\underline{v}) = \mu_{\hat{p}(\omega)}(\omega)$, it follows that there is a $\underline{v}' < \omega$ such that for all $\underline{v} \in (\underline{v}', \omega)$, the integrand in

$$\frac{\partial \Psi(q, p, \underline{v})}{\partial p}\bigg|_{(q,p)=(\hat{q}(\underline{v}),\hat{p}(\underline{v}))}$$

is bounded. Since $\lim_{\underline{v}\to\omega} \mu_{\hat{p}(\underline{v})}(\hat{q}(\underline{v})) = \omega$, the Dominated Convergence Theorem implies that (A.29) holds. Consequently, there is a $\underline{v}'' < \omega$ such that for all $\underline{v} \in (\underline{v}'', \omega)$, the determinant $|D_{q,p}\Phi(q, p, \underline{v})|$ is not zero at $(\hat{q}(\underline{v}), \hat{p}(\underline{v}))$.

Thus, the Implicit Function Theorem (see, e.g., Munkres, 1991, Thms. 9.1 and 9.2) applies: For every $\underline{v}''' \in (\underline{v}'', \omega)$, there are a neighborhood $V$ of $\underline{v}'''$ and unique continuous functions $\tilde{q}, \tilde{p}$ on $V$ such that $(\tilde{q}(\underline{v}'''), \tilde{p}(\underline{v}''')) = (\hat{q}(\underline{v}'''), \hat{p}(\underline{v}'''))$ and $\Phi(\tilde{q}(\underline{v}), \tilde{p}(\underline{v}), \underline{v}) = 0$ for all $\underline{v} \in V$. Moreover, the derivatives of $\tilde{q}$ and $\tilde{p}$ exist at all $\underline{v} \in V$ and are given by

$$\begin{pmatrix} \frac{d\tilde{q}(\underline{v})}{d\underline{v}} \\ \frac{d\tilde{p}(\underline{v})}{d\underline{v}} \end{pmatrix} = -[D_{q,p}\Phi(q, p, \underline{v})]^{-1} D_{\underline{v}}\Phi(q, p, \underline{v})\bigg|_{(q,p)=(\tilde{q}(\underline{v}),\tilde{p}(\underline{v}))}.$$

Hence, using (A.27) and (A.28),

$$\frac{d\tilde{p}(\underline{v})}{d\underline{v}} = -\frac{\frac{\partial^2\Psi(q,p,\underline{v})}{\partial q^2}\left[f(\underline{v})p + \frac{\partial\Psi(q,p,\underline{v})}{\partial \underline{v}}\right] - \frac{\partial\Psi(q,p,\underline{v})}{\partial q}\frac{\partial^2\Psi(q,p,\underline{v})}{\partial q\partial \underline{v}}}{|D_{q,p}\Phi(q, p, \underline{v})|}\Bigg|_{(q,p)=(\tilde{q}(\underline{v}),\tilde{p}(\underline{v}))},$$

which, noting that $\partial\Psi(q, p, \underline{v})/\partial q = 0$ at $(q, p) = (\tilde{q}(\underline{v}), \tilde{p}(\underline{v}))$, simplifies to

$$\frac{d\tilde{p}(\underline{v})}{d\underline{v}} = \frac{f(\underline{v})p + \frac{\partial\Psi(q,p,\underline{v})}{\partial \underline{v}}}{1 - F(\underline{v}) - \frac{\partial\Psi(q,p,\underline{v})}{\partial p}}\Bigg|_{(q,p)=(\tilde{q}(\underline{v}),\tilde{p}(\underline{v}))}. \tag{A.30}$$

As the function $\tilde{p}$ is unique, we have $d\hat{p}(\underline{v})/d\underline{v} = d\tilde{p}(\underline{v})/d\underline{v}$ at every $\underline{v} \in (\underline{v}''', \omega)$ where $d\hat{p}(\underline{v})/d\underline{v}$ exists. Observe that

$$\frac{\partial\Psi(q, p, \underline{v})}{\partial \underline{v}} = -x_q(\underline{v}, \mu_p(\underline{v}))qf(\underline{v}) = -\frac{p - \underline{v}}{\mu_p(\underline{v}) - p}\frac{\mu_p(\underline{v}) - q}{q - \underline{v}}qf(\underline{v}).$$

Since $\hat{q}(\underline{v})$ converges to $\mu_{\hat{p}(\omega)}(\omega)$, we obtain

$$\lim_{\underline{v}\to\omega} \frac{\partial\Psi(q,p,\underline{v})}{\partial\underline{v}}\bigg|_{(q,p)=(\hat{q}(\underline{v}),\hat{p}(\underline{v}))} = 0. \tag{A.31}$$

Finally, combining (A.30) with (A.31) and (A.29) thus yields (A.26). □

Now, by Claim A4 there exist $\underline{v}' < \omega$ such that

$$\frac{d\hat{p}(\underline{v})}{d\underline{v}} > \frac{f(\underline{v})[\hat{p}(\underline{v}') - \underline{v}]}{1 - F(\omega)} \quad \text{for all } \underline{v} \in (\underline{v}', \omega). \tag{A.32}$$

As $\hat{p}$ is increasing,

$$\hat{p}(\omega) - \hat{p}(\underline{v}') \geq \int_{\underline{v}'}^{\omega} \frac{d\hat{p}(\underline{v})}{d\underline{v}} d\underline{v} > \frac{1}{1-F(\omega)} \int_{\underline{v}'}^{\omega} [\hat{p}(\underline{v}') - v] dF(v),$$

where the second inequality follows from (A.32). As a consequence,

$$\int_{\underline{v}'}^{1} [v-\hat{p}(\underline{v}')]dF(v) - \int_{\omega}^{1} [v-\hat{p}(\omega)]dF(v) = \int_{\underline{v}'}^{\omega} [v-\hat{p}(\underline{v}')]dF(v) + [1-F(\omega)][\hat{p}(\omega)-\hat{p}(\underline{v}')] > 0,$$

that is, $(\hat{p}(\underline{v}'), \underline{v}')$ implements a strictly higher buyer payoff than $(\hat{p}(\omega), \omega)$. This completes the proof that $(p, \underline{v}) = (\hat{p}(\omega), \omega)$ is not optimal. □

**Proof of Corollary 1.** Suppose $F$ is regular and $E[v|v \leq p^*] < \Pi^*$. We will show that the seller payoff is strictly greater than $\Pi^*$ under all buyer-optimal information structures. By Proposition 1, these information structures implement a price $p < p^*$.

By contradiction, suppose there is a buyer-optimal information structure $(S, (G_v))$ that implements seller payoff $\Pi^*$, buyer payoff $U$, and price $p < p^*$. Under $(S, (G_v))$, valuations below $p^*$ are not pooled with valuations above $p^*$ (i.e., $\int_{\{s \in S: F_s(p^*) \in (0,1)\}} d\bar{G}(s) = 0$) because otherwise, the seller could first extend to a $p$-pairwise information structure that yields the same payoff (Lemma 1) and then perform the $p^*$-optimal extension (Lemma 2) to obtain a payoff strictly greater than $\Pi^*$. Recall that $H(w) = \int_{\{s \in S: E[v|s] \leq w\}} d\bar{G}(s)$ is the CDF of posterior valuations under $(S, (G_v))$. As valuations below $p^*$ are not pooled with valuations above $p^*$, we can write the buyer payoff as

$$U = \int_{p}^{1} (v-p)dH(v) = \int_{p}^{p^*} (v-p)dH(v) + \int_{p^*}^{1} (v-p)dF(v) \geq \int_{p^*}^{1} (v-p)dF(v). \tag{A.33}$$

By Theorem 1 and Propositions 2 and 3, there is a $\underline{v} < \omega$ such that $(\hat{p}(\underline{v}), \underline{v})$ is optimal, $\hat{p}(\underline{v}) = p$, and the implemented buyer payoff is

$$U = \int_{\mu_{\hat{p}(\underline{v})}(\underline{v})}^{1} (v - \hat{p}(\underline{v})) dF(v). \tag{A.34}$$

As shown at the beginning of the proof of Claim A4 (in the proof of Proposition 3), for any $\underline{v} < \omega$, the seller's extension payoff $\Psi(\cdot, \hat{p}(\underline{v}), \underline{v})$ is maximized at a unique $q \in (\underline{v}, \mu_{\hat{p}(\underline{v})}(\underline{v}))$. As the third derivative of $\Psi(\cdot, \hat{p}(\underline{v}), \underline{v})$ is negative (Lemma 5), $\Psi(\cdot, \hat{p}(\underline{v}), \underline{v})$ thus is decreasing at $\mu_{\hat{p}(\underline{v})}(\underline{v})$. Lemma 5 and the strict quasiconcavity of $[1-F(q)]q$ then imply $\mu_{\hat{p}(\underline{v})}(\underline{v}) > p^*$. Consequently, (A.34) yields

$$U < \int_{p^*}^{1} (v - p) dF(v),$$

which contradicts (A.33). Hence, there can be no such $(S, (G_v))$. □

**Proof of Proposition 4.** Fix $(S^a, (G_v^a))$ and an arbitrary incentive-compatible and individually rational menu $\{(X(\tilde{s}^a), c(\tilde{s}^a), p(\tilde{s}^a)) : \tilde{s}^a \in S^a\}$. We will construct a posted-price mechanism that yields the same buyer payoff and a weakly higher seller payoff.

First, we construct for each $\tilde{s}^a \in S^a$ the extension $X^I(\tilde{s}^a) = (\{BUY, NOT\}, (\hat{G}^b_{v,s^a}))$ from the extension $X(\tilde{s}^a) = (S^b, (G^b_{v,s^a}))$ as follows: Whenever $X(\tilde{s}^a)$ displays a signal $s^b \in S^b$ such that $E[v|s^a, s^b] \geq p(\tilde{s}^a)$ (i.e., the buyer buys), $X^I(\tilde{s}^a)$ displays the signal $BUY$. Otherwise, $X^I(\tilde{s}^a)$ displays the signal $NOT$. We write $\Pr[BUY|s^a]$ and $\Pr[NOT|s^a] = 1 - \Pr[BUY|s^a]$ for the respective probabilities. Clearly, the expected buyer payoff conditional on $s^a$ from $(X^I(\tilde{s}^a), c(\tilde{s}^a), p(\tilde{s}^a))$ and from $(X(\tilde{s}^a), c(\tilde{s}^a), p(\tilde{s}^a))$ are the same, and the buyer buys if and only if he observes signal $BUY$. Hence, the "modified" menu $\{(X^I(\tilde{s}^a), c(\tilde{s}^a), p(\tilde{s}^a)) : \tilde{s}^a \in S^a\}$ is incentive compatible and individually rational and yields the same buyer payoff and seller payoff as the original menu.

Let $p := \inf_{\tilde{s}^a \in S^a} c(\tilde{s}^a) + p(\tilde{s}^a)$. By deviating (and always buying), the buyer can secure himself a payoff arbitrarily close to $E[v|s^a] - p$. Incentive compatibility of the modified menu thus implies

$$\Pr[BUY|s^a][E[v|s^a, BUY] - p(s^a)] - c(s^a) \geq E[v|s^a] - p. \tag{A.35}$$

Now, consider the posted-price mechanism $(X^{II}, 0, p)$, where for each $(v, s^a)$ the CDF $G^b_{v,s^a}$ of $X^{II} = (\{BUY, NOT\}, (G^b_{v,s^a}))$ coincides with the respective CDF of $X^I(s^a)$. We will compare purchase decision and buyer payoff conditional on $s^a$ under $(X^{II}, 0, p)$ with those under the modified menu. For $s^a$ such that $(c(s^a), p(s^a)) = (0, p)$, it is immediate that they are the same. In what follows, we hence assume $(c(s^a), p(s^a)) \neq (0, p)$.

For $s^a$ such that $\Pr[BUY|s^a] = 0$, individual rationality of the modified menu implies $c(s^a) = 0$ and, as $(c(s^a), p(s^a)) \neq (0, p)$, (A.35) yields $0 > E[v|s^a] - p$. Under $(X^{II}, 0, p)$, the buyer hence also never buys and gets the same payoff as under the modified menu.

For $s^a$ such that $\Pr[BUY|s^a] = 1$, the definition of $p$ implies that (A.35) holds with equality. Under $(X^{II}, 0, p)$, the buyer hence also always buys and gets the same payoff as under the modified menu.

For $s^a$ such that $\Pr[BUY|s^a] \in (0, 1)$, we use

$$E[v|s^a] = \Pr[BUY|s^a]E[v|s^a, BUY] + \Pr[NOT|s^a]E[v|s^a, NOT]$$

(i.e., the martingale property of posterior valuations) to rewrite inequality (A.35) as

$$\Pr[NOT|s^a](p - E[v|s^a, NOT]) \geq \Pr[BUY|s^a][p(s^a) - p] + c(s^a).$$

As $(c(s^a), p(s^a)) \neq (0, p)$, the right-hand side is strictly positive. So, $p > E[v|s^a, NOT]$. On the other hand, because the modified menu is individually rational,

$$\Pr[BUY|s^a][E[v|s^a, BUY] - p(s^a)] - c(s^a) \geq 0.$$

As $(c(s^a), p(s^a)) \neq (0, p)$, this implies $E[v|s^a, BUY] > p$. Under $(X^{II}, 0, p)$, the buyer hence also buys if and only if he observes the signal $BUY$. Since $(c(s^a), p(s^a)) \neq (0, p)$, however, the buyer payoff is strictly higher than under the modified menu,

$$\Pr[BUY|s^a](E[v|s^a, BUY] - p) > \Pr[BUY|s^a][E[v|s^a, BUY] - p(s^a)] - c(s^a).$$

Finally, we construct an extension $X^{III}$ from $X^{II}$ such that $(X^{III}, 0, p)$ yields the same buyer payoff as the modified menu for all $s^a$. $X^{III}$ coincides with $X^{II}$ unless $s^a$ is such that $\Pr[BUY|s^a] \in (0, 1)$ and $(c(s^a), p(s^a)) \neq (0, p)$. In the latter case, $X^{III}$ is defined as follows: Whenever $X^{II}$ displays $BUY$ conditional on $s^a$, $X^{III}$ displays $BUY$. Whenever $X^{II}$ displays $NOT$ conditional on $s^a$, $X^{III}$ displays $BUY$ with probability $\epsilon$ and $NOT$ with probability $1 - \epsilon$. Before specifying $\epsilon$, note that the buyer payoff is continuous and decreasing as a function of $\epsilon$. At $\epsilon = 0$ (same information), it equals the payoff under the posted-price mechanism $(X^{II}, 0, p)$. At $\epsilon = 1$ (no information), it is weakly smaller than the buyer payoff under the modified menu by (A.35). Hence, there exists an $\epsilon > 0$ such that the payoff is equal to the payoff under the modified menu and $E[v|s^a, BUY] \geq p$. Choose this $\epsilon$. By construction, the probability of trade under $(X^{III}, 0, p)$ is higher than under the modified menu, resulting in a higher social surplus.

We conclude that compared with $\{(X(\tilde{s}^a), c(\tilde{s}^a), p(\tilde{s}^a)) : \tilde{s}^a \in S^a\}$, the posted-price mechanism $(X^{III}, 0, p)$ yields the same buyer payoff and a weakly higher social surplus, which implies a weakly higher seller payoff. □

**Proof of Proposition 5.** The "only if" part is clear. It remains to prove that if a $p$-pairwise information structure is not extensionproof, then it is also not weakly extensionproof. We will do so by showing that, for every $p$-pairwise information structure and every $q > p$, there is an independent extension that induces the same probability of trade at price $q$, and hence the same seller payoff, as the $q$-optimal extension.

Let $(S^a, (G^a_v))$ be $p$-pairwise and suppose the seller sets price $q > p$. Regardless of the extension, there is trade with probability one after all signals $s^a = (v_L, v_H)$ with $v_L = v_H \geq q$, and there is no trade after all signals $s^a$ with $v_H < q$. Only after signals $s^a$ with $v_L < q \leq v_H$, the probability of trade depends on the extension. We denote the set of such signals by $\widehat{S}^a := \{s^a \in S^a : v_L < q \leq v_H\}$.

Consider the independent extension that, conditional on $v \notin [p, q)$, draws the additional signal $s^b$ from $S^b = [q, 1]$ according to $G^b_v$. The CDFs $G^b_v$ are defined as follows:

- If $v \in [0, p)$, then $G^b_v$ has support $[q, 1]$ and
$$G^b_v(s^b) = \frac{p - v}{q - v}\frac{s^b - q}{s^b - p} + 1 - \frac{p - v}{q - v}\frac{1 - q}{1 - p}.$$
Hence, $G^b_v$ has an atom at $s^b = q$ and for $s^b \in (q, 1]$ a PDF $g^b_v$ with
$$g^b_v(s^b) = \frac{p - v}{q - v}\frac{q - p}{(s^b - p)^2}.$$

- If $v \in [q, 1]$, then $G^b_v$ has support $[q, v]$ and
$$G^b_v(s^b) = \frac{v - p}{v - q}\frac{s^b - q}{s^b - p}.$$

Hence, $G_v^b$ is atomless and has a PDF $g_v^b$ with

$$g_v^b(s^b) = \frac{v-p}{v-q}\frac{q-p}{(s^b-p)^2}.$$

Now, suppose the buyer observes $s^a = (v_L, v_H) \in \widehat{S}^a$ and $s^b$. If $s^b \notin (q, v_H]$, this perfectly reveals that the valuation is $v = v_L$. If $s^b \in (q, v_H]$, the posterior valuation is

$$\begin{aligned}
E[v|s^a, s^b] &= \frac{F_{s^a}(v_L)\, g_{v_L}^b(s^b)\, v_L + [1 - F_{s^a}(v_L)]\, g_{v_H}^b(s^b)\, v_H}{F_{s^a}(v_L)\, g_{v_L}^b(s^b) + [1 - F_{s^a}(v_L)]\, g_{v_H}^b(s^b)} \\
&= \frac{F_{s^a}(v_L)\frac{p-v_L}{v_H-p}\frac{v_H-q}{q-v_L}v_L + [1 - F_{s^a}(v_L)]v_H}{F_{s^a}(v_L)\frac{p-v_L}{v_H-p}\frac{v_H-q}{q-v_L} + 1 - F_{s^a}(v_L)} \\
&= \frac{F_{s^a}(v_L)x_q(v_L, v_H)v_L + [1 - F_{s^a}(v_L)]v_H}{F_{s^a}(v_L)x_q(v_L, v_H) + 1 - F_{s^a}(v_L)} \quad = q.
\end{aligned}$$

Conditional on $s^a \in \widehat{S}^a$, the probability of trade is thus equal to the probability of $s^b \in (q, v_H]$. Hence, conditional on $s^a \in \widehat{S}^a$ and $v = v_H$, the probability of trade is one, and conditional on $s^a \in \widehat{S}^a$ and $v = v_L$, the probability of trade is equal to

$$G_{v_L}^b(v_H) - G_{v_L}^b(q) = \frac{p-v_L}{v_H-p}\frac{v_H-q}{q-v_L} = x_q(v_L, v_H).$$

By Lemma 2, the probability of trade is the same as under the $q$-optimal extension. □

**Proof of Lemma 6.** Consider an information structure $(S, (G_v))$ that induces buyer payoff $\overline{U}$ and suppose (15) does not hold, that is, $\int_{\{s \in S: F_s(p^*) \in (0,1)\}} d\bar{G}(s) > 0$. By the right-continuity of distribution functions, there exists a $\delta > 0$ and a subset of signals

$$M := \{s \in S : 0 < F_s(p^*) \text{ and } F_s(p^* + \delta) < 1\}$$

such that $\int_M d\bar{G}(s) > 0$. To show that $(S, (G_v))$ is not weakly extensionproof, we will construct an independent extension under which the probability of trade at price $p^*$ is strictly greater than $1 - F(p^*)$, implying a seller payoff strictly greater than $\Pi^*$.

Consider the following independent extension. If $v \in (p^*, p^* + \delta]$, display a signal $BUY1$ with probability one. If $v > p^* + \delta$, display a signal $BUY2$ with probability one. If $v \leq p^*$, display $BUY2$ with some probability $\epsilon > 0$ and otherwise a signal $NOT$. Hence, for $s \in S$, the posterior valuation given $s$ and $BUY1$ is greater than $p^*$, and the posterior valuation given $s$ and $BUY2$ is[22]

$$w_\epsilon(s, BUY2) := \frac{\epsilon F_s(p^*)}{\epsilon F_s(p^*) + 1 - F_s(p^* + \delta)} E[v|s, v \leq p^*] + \frac{1 - F_s(p^* + \delta)}{\epsilon F_s(p^*) + 1 - F_s(p^* + \delta)} E[v|s, v > p^* + \delta].$$

Let $M_\epsilon := \{s \in M : w_\epsilon(s, BUY2) < p^*\}$.

Under this extension, the probability of trade at price $p^*$ is

$$Q := \underbrace{\epsilon \int_0^{p^*} \int_{M \setminus M_\epsilon} dG_v(s) dF(v)}_{\text{(i)}} + \underbrace{\int_{p^*}^{p^*+\delta} dF(v)}_{\text{(ii)}} + \underbrace{\int_{p^*+\delta}^{1} \int_{S \setminus M_\epsilon} dG_v(s) dF(v)}_{\text{(iii)}}.$$

Part (i) covers the case that $v \in [0, p^*]$, $BUY2$ is displayed, and $w_\epsilon(s, BUY2) \geq p^*$. The latter holds if and only if $s \in M \setminus M_\epsilon$ because for $s \notin M$, as $v$ is in the support of $F_s$ for almost all $s$, $F_s(p^* + \delta) = 1$ and thus $w_\epsilon(s, BUY2) < p^*$. Part (ii) represents the buyer buying after observing $BUY1$. Part (iii) covers the case that $v \in (p^* + \delta, 1]$ and $w_\epsilon(s, BUY2) \geq p^*$. The latter also holds for $s \notin M$ because then $F_s(p^*) = 0$ as $v$ is in the support of $F_s$ for almost all $s$. The probability $Q$ can be restated as

$$Q = \left[\int_0^{p^*} \int_M dG_v(s) dF(v) - \int_0^{p^*} \int_{M_\epsilon} dG_v(s) dF(v)\right] \epsilon + 1 - F(p^*) - \int_{p^*+\delta}^{1} \int_{M_\epsilon} dG_v(s) dF(v),$$

which, by the definition of the posterior in (1), is equivalent to

$$Q = \left[\int_M F_s(p^*) d\bar{G}(s) - \int_{M_\epsilon} F_s(p^*) d\bar{G}(s)\right] \epsilon + 1 - F(p^*) - \int_{M_\epsilon} [1 - F_s(p^* + \delta)] d\bar{G}(s). \quad \text{(A.36)}$$

By the definition of $M_\epsilon$, $E[w_\epsilon(s, BUY2)|s \in M_\epsilon]$ is equal to

$$\frac{\epsilon \int_{M_\epsilon} F_s(p^*) d\bar{G}(s)}{\epsilon \int_{M_\epsilon} F_s(p^*) d\bar{G}(s) + \int_{M_\epsilon} (1 - F_s(p^* + \delta)) d\bar{G}(s)} E[v|s \in M_\epsilon, v \leq p^*] + \frac{\int_{M_\epsilon} (1 - F_s(p^* + \delta)) d\bar{G}(s)}{\epsilon \int_{M_\epsilon} F_s(p^*) d\bar{G}(s) + \int_{M_\epsilon} (1 - F_s(p^* + \delta)) d\bar{G}(s)} E[v|s \in M_\epsilon, v > p^* + \delta]. \quad \text{(A.37)}$$

[22]As $BUY2$ is displayed only if $v \notin (p^*, p^* + \delta]$ and as the true valuation $v$ is in the support of the posterior distribution $F_s$ for almost all $s$, the extended information structure assigns probability zero to signals $(s, BUY2)$ such that $F_s(p^*) = 1 - F_s(p^* + \delta) = 0$.

Using (A.37), we have

$$
\begin{aligned}
& E[w_\epsilon(s, BUY2)|s \in M_\epsilon] < p^* \\
\iff \quad & \int_{M_\epsilon} [1 - F_s(p^* + \delta)] d\bar{G}(s) < \int_{M_\epsilon} F_s(p^*) d\bar{G}(s) \frac{p^* - E[v|s \in M_\epsilon, v \leq p^*]}{E[v|s \in M_\epsilon, v > p^* + \delta] - p^*} \epsilon \\
\implies \quad & \int_{M_\epsilon} [1 - F_s(p^* + \delta)] d\bar{G}(s) < \int_{M_\epsilon} F_s(p^*) d\bar{G}(s) \frac{p^*}{\delta} \epsilon. \qquad \text{(A.38)}
\end{aligned}
$$

Combining (A.38) with (A.36) gives

$$
Q > 1 - F(p^*) + \left[ \int_M F_s(p^*) d\bar{G}(s) - \int_{M_\epsilon} F_s(p^*) d\bar{G}(s) \left( 1 + \frac{p^*}{\delta} \right) \right] \epsilon. \qquad \text{(A.39)}
$$

Now, $\lim_{\epsilon \to 0} w_\epsilon(s, BUY2) = E[v|s, v > p^* + \delta] > p^*$. By Ergorov's Theorem, the convergence is uniform across $s$ except on a subset of arbitrarily small measure. Hence,

$$
\lim_{\epsilon \to 0} \int_{M_\epsilon} F_s(p^*) d\bar{G}(s) = 0.
$$

On the other hand, $\int_M F_s(p^*) d\bar{G}(s) > 0$. Hence, there exist $\epsilon > 0$ such that

$$
\int_M F_s(p^*) d\bar{G}(s) - \int_{M_\epsilon} F_s(p^*) d\bar{G}(s) \left( 1 + \frac{p^*}{\delta} \right) > 0.
$$

For such $\epsilon > 0$, by (A.39), the probability of trade at $p^*$ is $Q > 1 - F(p^*)$. □

**Proof of Proposition 6.** Consider any weakly extensionproof information structure $(S, (G_v))$ that induces buyer payoff $\overline{U}$, which means trade with probability one at price $\Pi^*$. Then, the CDF of posterior valuations $H$ satisfies $H(w) = 0$ for all $w < \Pi^*$ and so

$$
\int_0^{p^*} w dH(w) \geq H(p^*) \Pi^*.
$$

By Lemma 6, this inequality can be restated as

$$
\begin{aligned}
& \int_{\{s \in S : F_s(p^*) = 1\}} \int_0^1 v dF_s(v) d\bar{G}(s) \geq \int_{\{s \in S : F_s(p^*) = 1\}} \int_0^1 dF_s(v) d\bar{G}(s) \Pi^* \\
\iff \quad & \int_S \int_0^{p^*} v dF_s(v) d\bar{G}(s) \geq \int_S \int_0^{p^*} dF_s(v) d\bar{G}(s) \Pi^*.
\end{aligned}
$$

Using the definition of the posterior in (1), this is equivalent to

$$
\int_0^{p^*} v dF(v) \geq F(p^*) \Pi^* \quad \iff \quad E[v|v \leq p^*] \geq \Pi^*.
$$

□

**Proof of Proposition 7.** According to Roesler and Szentes (2017, Corollary 1), $p^{RS} \leq \Pi^*$. Our proof consists in showing that this inequality is strict. Consequently, any information structure that is buyer optimal in the setting of Roesler and Szentes is not weakly extensionproof: the seller can increase her payoff from $p^{RS} < \Pi^*$ to $\Pi^*$ by independently extending to a perfect information structure.

We start with an auxiliary result. According to Roesler and Szentes (2017, Lemma 1), there is a unique $B^* \in [\Pi^*, 1]$ such that $F$ is a mean-preserving spread of $H_{\Pi^*}^{B^*}$. With the following claim, we strengthen this to $B^* \in (\Pi^*, 1)$.

**Claim A5.** *There is a unique $B^* \in (\Pi^*, 1)$ such that $F$ and $H_{\Pi^*}^{B^*}$ have the same mean.*

*Proof.* Note that $\Pi^* = \max_p[1 - F(p)]p$ implies $0 < \Pi^* < \int_0^1 v dF(v)$.

As $[1 - F(w)]w \leq \Pi^*$, we have $1 - \frac{\Pi^*}{w} \leq F(w)$. Hence $H_{\Pi^*}^1(w) \leq F(w)$ for all $w \in [0, 1]$, that is, $H_{\Pi^*}^1$ first-order stochastically dominates $F$. The dominance is strict since $H_{\Pi^*}^1(w) = 0 < F(w)$ for all $w \in (0, \Pi^*]$. Therefore

$$\int_0^1 w dH_{\Pi^*}^1(w) > \int_0^1 v dF(v) > \Pi^* = \int_0^1 w dH_{\Pi^*}^{\Pi^*}(w).$$

As $\int_0^1 w dH_{\Pi^*}^B(w)$ is continuous and strictly increasing in $B$, there must be a unique $B^* \in (\Pi^*, 1)$ such that $F$ and $H_{\Pi^*}^{B^*}$ have the same mean, $\int_0^1 w dH_{\Pi^*}^{B^*}(w) = \int_0^1 v dF(v)$. □

Now, since $F$ is a mean-preserving spread of $H_{\Pi^*}^{B^*}$,

$$\int_0^w F(z)dz \geq \int_0^w H_{\Pi^*}^{B^*}(z)dz \quad \text{for all } w \in [0, 1], \text{ with equality for } w = 1.$$

We next show that the above inequality is strict for all $w \in (0, 1)$.

**Claim A6.** *$\int_0^w F(z)dz > \int_0^w H_{\Pi^*}^{B^*}(z)dz$ for all $w \in (0, 1)$.*

*Proof.* Define $\Gamma(w) := \int_0^w [F(z) - H_{\Pi^*}^{B^*}(z)]dz$. We have to prove that $\Gamma(w) > 0$ for all $w \in (0, 1)$. For $w \in (0, \Pi^*]$, $\Gamma(w) = \int_0^w F(z)dz > 0$. For $w \in [\Pi^*, B^*)$, $F(w) - H_{\Pi^*}^{B^*}(w)$ is continuous, and so we can differentiate $\Gamma$ to get

$$\Gamma'(w) = F(w) - H_{\Pi^*}^{B^*}(w) = \frac{\Pi^* - w[1 - F(w)]}{w} \geq 0,$$

where the inequality holds since $\Pi^* = \max_p[1 - F(p)]p$. Therefore, $\Gamma(w) > 0$ also for $w \in (\Pi^*, B^*)$. For $w \in [B^*, 1)$,

$$\Gamma(w) = \int_0^w [F(z) - H_{\Pi^*}^{B^*}(z)]dz = -\int_w^1 [F(z) - H_{\Pi^*}^{B^*}(z)]dz = \int_w^1 [1 - F(z)]dz > 0,$$

where the second equality holds because $F$ and $H_{\Pi^*}^{B^*}$ have the same mean. □

Recall that $p^{RS}$ is the smallest price $q$ for which there exists $B \in [q, 1]$ such that $F$ is a mean-preserving spread of $H_q^B$. The following claim implies $p^{RS} < \Pi^*$, which completes the proof of the proposition.

**Claim A7.** *There exist $q < \Pi^*$ and $B \in [q, 1]$ such that $F$ is a mean-preserving spread of $H_q^B$*

*Proof.* Take any $B \in (B^*, 1]$, which exists by Claim A5. For $q \leq \Pi^*$, $\int_0^1 H_q^B(w)dw$ is strictly decreasing in $q$. By the Dominated Convergence Theorem, $\int_0^1 H_q^B(w)dw$ is furthermore continuous in $q$. As $\int_0^1 H_0^B(w)dw = 1 > \int_0^1 H_{\Pi^*}^{B^*}(w)dw$ and $\int_0^1 H_{\Pi^*}^B(w)dw < \int_0^1 H_{\Pi^*}^{B^*}(w)dw$, it follows that there is a unique $q(B) \in (0, \Pi^*)$ such that

$$\int_0^1 H_{q(B)}^B(w)dw = \int_0^1 H_{\Pi^*}^{B^*}(w)dw = \int_0^1 F(w)dw. \tag{A.40}$$

For every $w \in [0, 1]$ and every sequence of values $B \in (B^*, 1]$, since $\lim_{B \to B^*} q(B) = \Pi^*$, the Dominated Convergence Theorem gives

$$\lim_{B \to B^*} \int_0^w H_{q(B)}^B(z)dz = \int_0^w H_{\Pi^*}^{B^*}(z)dz.$$

Choose $B', B'' \in (B^*, 1]$ such that $B' < B''$. Since $q(B') > q(B'')$, we have for $w \in [0, B')$,

$$\int_0^w H_{q(B')}^{B'}(z)dz \leq \int_0^w H_{q(B'')}^{B''}(z)dz.$$

Similarly, for $w \in [B', 1]$

$$\int_0^w [H_{q(B')}^{B'}(z) - H_{q(B'')}^{B''}(z)]dz = \int_w^1 [H_{q(B'')}^{B''}(z) - H_{q(B')}^{B'}(z)]dz = \int_w^1 [H_{q(B'')}^{B''}(z) - 1]dz \leq 0.$$

So for every $w \in [0, 1]$, the sequence of integrals $\int_0^w H_{q(B)}^B(z)dz$ is increasing. By Dini's Theorem, the convergence of $\int_0^w H_{q(B)}^B(z)dz$ to $\int_0^w H_{\Pi^*}^{B^*}(z)dz$ is thus uniform across $w$.

Claim A6 and the uniform convergence imply that there exists $\widehat{B} \in (B^*, 1]$ such that

$$\int_0^w H^{\widehat{B}}_{q(\widehat{B})}(z)dz - \int_0^w H^{B^*}_{\Pi^*}(z)dz < \int_0^w F(z)dz - \int_0^w H^{B^*}_{\Pi^*}(z)dz \quad \text{for all } w \in (0,1).$$

By (A.40), $F$ is thus a mean-preserving spread of $H^{\widehat{B}}_{q(\widehat{B})}$, and $q(\widehat{B}) < \Pi^*$. □□

**Proof of Proposition 8.** According to Roesler and Szentes (2017, Lemma 1), there exists a unique $B^*$ such that $F$ is a mean-preserving spread of $H^{B^*}_{\Pi^*}$. The information structures in the RS class that induce buyer payoff $\overline{U}$ are thus all $(S, (G_v))$ that induce the CDF of posterior valuations $H^{B^*}_{\Pi^*}$. Consider any such $(S, (G_v))$. We will show that $\int_{\{s\in S: F_s(p^*)\in(0,1)\}} d\bar{G}(s) > 0$. By Lemma 6, this implies that $(S, (G_v))$ is not weakly extensionproof and, accordingly, also not extensionproof.

By contradiction, suppose $\int_{\{s\in S: F_s(p^*)\in(0,1)\}} d\bar{G}(s) = 0$. Then,

$$\begin{aligned} \int_{p^*}^1 w dH^{B^*}_{\Pi^*}(w) &= \int_{\{s\in S: F_s(p^*)=0\}} \int_0^1 v dF_s(v) d\bar{G}(s) \\ &= \int_S \int_{p^*}^1 v dF_s(v) d\bar{G}(s) = \int_{p^*}^1 v dF(v), \end{aligned} \tag{A.41}$$

where the last equality follows from the definition of the posterior in (1). We consider two cases. First, suppose $B^* \leq p^*$. As $H^{B^*}_{\Pi^*}(w) = 1$ for all $w \geq B^*$, we have a contradiction to (A.41). Second, suppose $B^* > p^*$. By the definition of the RS class, $[1 - H^{B^*}_{\Pi^*}(p)]p = \Pi^*$ for all $p \in [\Pi^*, B^*]$. On the other hand, $[1 - F(p)]p < \Pi^*$ for all $p < p^*$. Consequently, $H^{B^*}_{\Pi^*}(p) < F(p)$ for all $p \in (0, p^*)$, whereas $H^{B^*}_{\Pi^*}(p^*) = F(p^*)$. We thus have $\int_0^{p^*} w dH^{B^*}_{\Pi^*}(w) > \int_0^{p^*} v dF(v)$. Given (A.41), this implies that $F$ is not a mean-preserving spread of $H^{B^*}_{\Pi^*}$; a contradiction. □

**Proof of Lemma 7.** Let $(S^a, (G^a_v))$ be a buyer-extensionproof information structure that induces price $p$ and a seller payoff weakly greater than $\Pi^*$. We proceed as in the proof of Lemma 1 and extend $(S^a, (G^a_v))$ to the $p$-pairwise information structure $(S^{abc}, (G^{abc}_v))$. Compared with those under $(S^a, (G^a_v))$, the buyer payoff and the seller payoff at price $p$ under $(S^{abc}, (G^{abc}_v))$ either both remain unchanged or both strictly increase. (They remain unchanged if and only if

$$\int_{\{v\in[0,1]: v>p\}} \int_{\{s^a\in S^a: E[v|s^a]<p\}} dG^a_v(s^a) dF(v) = 0,$$

that is, if and only if under $(S^a, (G^a_v))$ no valuations $v > p$ are pooled into posterior valuations $E[v|s^a] < p$.) Because $(S^a, (G^a_v))$ is buyer extensionproof, we can rule out, first, that $(S^{abc}, (G^{abc}_v))$ induces price $p$ and the payoffs strictly increase, and, second, that $(S^{abc}, (G^{abc}_v))$ induces some price $q < p$. Now, under every $p$-pairwise information structure, $H(q) \geq F(q)$ and thus $[1 - H(q)]q \leq [1 - F(q)]q \leq \Pi^*$ for all $q > p$. Since under $(S^a, (G^a_v))$ the seller payoff at price $p$ is weakly greater than $\Pi^*$ by assumption, it follows that $(S^{abc}, (G^{abc}_v))$ induces price $p$. □

**Proof of Proposition 9.** In the main text. □

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