α - Maxmin Expected Utility with Non-Unique Prior

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Abstract

Preferences over acts have an α-Maxmin Expected Utility (α-MEU) representation if they can be represented by the α-mixture of the worst and the best expected utility over a set of priors. The case $\alpha = 1$ is characterized in Gilboa and Schmeidler (1989).

In this paper I provide an axiomatic characterization for all $\alpha \in [0,1]\backslash\{\frac{1}{2}\}$. The introduced axioms are of a novel type and so is the proof technique. The characterization allows tracing back the phenomenon of multiplicity of representation, which is inherent in the α-MEU model, to preference conditions.

Keywords: α-maxmin expected utility; Ambiguity

1 Introduction

The main result of this paper is an axiomatic characterization of α-MEU preferences for all $\alpha \in [0,1]\backslash\{\frac{1}{2}\}$ in the standard framework of Anscombe and Aumann (1963).[1] Theorem 1 is the first axiomatization of this kind and allows a deeper understanding of the well-known phenomenon of multiplicity of representation inherent in the model, hereby

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[1] There is a significant theoretical literature on the α-MEU model (Chateauneuf, Eichberger, and Grant (2007), Frick, Iijima, and Yaouanq (2020), Ghirardato, Maccheroni, and Marinacci (2004), Gul and Pesendorfer (2015) Hill (2019a), Klibanoff, Mukerji, Seo, and Stanca (2021), Kopylov (2003), Kopylov (2016), Olszewski (2007), Vierø (2009)); see the discussion in Section 5.

complementing results from Ghirardato, Maccheroni, and Marinacci (2002), Siniscalchi (2006) and Frick, Iijima, and Yaouanq (2020).

I achieve the characterization through a novel proof technique, hereby utilizing the existing representation of MEU preferences in Gilboa and Schmeidler (1989): I identify a link between the representation functionals of an α-MEU preference and its "pessimistic twin", these being characterized by the same utility function and prior set as well as $\alpha = 1$ for the pessimistic twin. This link is well-defined (see equation (24) in the Appendix) and allows me to deduce the exact properties of an α-MEU functional. The deduced properties permit the derivation of the correct axioms.

These axioms, *α-Monotonicity* and *α-Ambiguity Attitude*, state preference restrictions over mixtures of constant equivalents of acts. This is, as far as I am aware, a novel approach. The axioms furthermore rely on the concept of duo of complementary pairs, introduced in Section 3.

The α-MEU model is of much interest to the applied decision-theoretic literature as it can model deviations from the pure pessimism that the MEU model inherently embodies. The suggestion that decision makers are not purely pessimistic is supported by a vast amount of empirical literature, see Trautmann and Van De Kuilen (2015) for a review, making axiomatic characterizations of α-MEU relevant for applications.

The paper is organized as follows. In Section 2 I introduce the framework, notation and relevant known axioms. Section 3 is the core of the paper. I introduce the axioms as well as the representation result and provide a brief discussion of the proof technique. Section 4 provides a deeper discussion of the introduced axioms and illustrates the consequences on the phenomenon of multiplicity of representation inherent in the α-MEU model. In Section 5 I discuss related literature, shed further light on the type of axioms introduced and discuss the case $\alpha = \frac{1}{2}$. I end with a conclusion. All proofs are in the appendix.

2 Preliminaries

2.1 Framework

Just as in Gilboa and Schmeidler (1989), the framework of Anscombe and Aumann (1963) is considered. Let S be a nonempty *state space* and Σ an algebra on S. Elements of Σ are called *events*. The set of Σ-measurable, finitely additive probability measures on S is denoted by $\Delta(S, \Sigma)$. The weak* topology is assumed. Let X, the set of *consequences*, be the set of finite-support lotteries over a nonempty set Z. *Acts* are Σ-measurable mappings from S to X. With the usual abuse of notation, X also denotes the set of *constant acts*, i.e. acts which result in the same lottery in every state. A decision maker's *preferences* are modeled by a binary relation $\succcurlyeq$ over $\mathcal{F}$, the set of simple (i.e. finite-valued) acts. The asymmetric and symmetric parts of $\succcurlyeq$ are denoted by $\succ$ and $\sim$, respectively.

For an act $f \in \mathcal{F}$, a *constant equivalent* of f is a constant act $x \in X$ such that $x \sim f$. Such a constant equivalent is denoted by x_f. For $f, g \in \mathcal{F}$ and $\lambda \in [0,1]$, $\lambda f + (1-\lambda)g$ is the act which results in $\lambda f(s) + (1-\lambda)g(s) \in X$ for all $s \in S$.

A functional $V : \mathcal{F} \to \mathbb{R}$ is a *representation functional* of $\succcurlyeq$ if for all $f, g \in \mathcal{F}$, $f \succcurlyeq g$ if and only if $V(f) \geq V(g)$. A binary relation $\succcurlyeq$ on $\mathcal{F}$ has an α-Maxmin Expected Utility (α-*MEU*) representation if there exist an affine $u : X \to \mathbb{R}$, a nonempty, weak* compact and convex set of priors $\mathcal{C} \subseteq \Delta(S, \Sigma)$ and some $\alpha \in [0,1]$ such that $\succcurlyeq$ is represented by the functional $V : \mathcal{F} \to \mathbb{R}$ with

$$V(f) = \alpha \min_{P \in \mathcal{C}} \int u(f)\ dP + (1-\alpha) \max_{P \in \mathcal{C}} \int u(f)\ dP.$$

The evaluation of each act is thus the weighted utility sum of the worst and best scenario over a set of priors.

2.2 Axioms of Gilboa and Schmeidler (1989) and the existence of constant equivalents

The following axioms are the ones from Gilboa and Schmeidler (1989).

A.1. Weak Order. $\succcurlyeq$ is complete and transitive.

A.2. Certainty-Independence. For $f, g \in \mathcal{F}$, $x \in X$ and $\beta \in (0,1)$

$$f \succcurlyeq g \iff \beta f + (1-\beta)x \succcurlyeq \beta g + (1-\beta)x.$$

A.3. Continuity. For $f, g, h \in \mathcal{F}$ if $f \succ g \succ h$ then there exist $\lambda, \mu \in (0,1)$ such that $\lambda f + (1-\lambda)h \succ g \succ \mu f + (1-\mu)h$.

A.4. Monotonicity. For $f, g \in \mathcal{F}$, if $f(s) \succcurlyeq g(s)$ for all $s \in S$ then $f \succcurlyeq g$.

A.5. Uncertainty Aversion. For $f, g \in \mathcal{F}$ and $\beta \in [0,1]$ if $f \succcurlyeq g$ then $\beta f + (1-\beta)g \succcurlyeq g$.

A.6. Non-degeneracy. There are $f, g \in \mathcal{F}$ such that $f \succ g$.

Gilboa and Schmeidler (1989) show that a preference relation satisfies *A.1*-*A.5* if and only if it has an MEU (i.e. 1-MEU) representation. The prior set is unique if *A.6* added.

For later reference consider the following counterpart of $A.5$.

A.5'. Uncertainty Seeking. For $f, g \in \mathcal{F}$ and $\beta \in [0,1]$ if $f \succcurlyeq g$ then $f \succcurlyeq \beta f + (1-\beta)g$.

When $A.5$ is replaced by $A.5'$, a characterization of 0-MEU (also referred to as Maxmax Expected Utility) preferences is achieved.[2]

The following axiom states that every act is at least as good as its worst consequence and at most as good as its best consequence. It is a weakening of $A.4$.[3]

A.7. Internality. For $f \in \mathcal{F}$ and $x, y \in X$ if $x \succcurlyeq f(s) \succcurlyeq y$ for all $s \in S$ then $x \succcurlyeq f \succcurlyeq y$.

$A.7$ is needed to ensure that constant equivalents exist. Due to its importance for this work I state this (known[4]) result as a lemma and provide a proof in the appendix.

Lemma 1. *If $A.1$, $A.3$ and $A.7$ hold, constant equivalents exist.*

[2] See for instance Machina and Siniscalchi (2014), fn. 56).

[3] The axiom is not new and appears in different contexts in the theoretical literature, see Section 5 for details.

[4] See for instance Nascimento and Riella (2010), who use the name "constant monotonicity" for $A.7$.

3 Axioms, representation result and proof technique

In this section I introduce the axioms α-Monotonicity and α-Ambiguity Attitude and state the representation result. Subsequently I explain the main ideas of the proof technique.

3.1 $A.4(\alpha)$, $A.5(\alpha)$ and the representation result for α-MEU

My axioms rely on the concept of *duos of complementary pairs*, which in return relies on *complementary pairs* as introduced in Siniscalchi (2009).

Definition 1 (Complementary pairs (Siniscalchi (2009)))**.** *Two acts $f, f' \in \mathcal{F}$ are called complementary if $\frac{1}{2}f(s) + \frac{1}{2}f'(s) \sim \frac{1}{2}f(s') + \frac{1}{2}f'(s')$ for all $s, s' \in S$. If two acts $f, f' \in \mathcal{F}$ are complementary then (f, f') is called a complementary pair.*

Complementary acts provide perfect hedges against each other as their utility profiles are "mirror images" (Siniscalchi (2009), p. 810).

Definition 2 (Duo of complementary pairs)**.** *If (f, f') and (g, g') are complementary pairs such that $\frac{1}{2}f(s) + \frac{1}{2}f'(s) \sim \frac{1}{2}g(s) + \frac{1}{2}g'(s)$ for some (and thus for all) $s \in S$, then the tuple $(f, f'; g, g')$ is a duo of complementary pairs.*

For a duo of complementary pairs, the indifference condition also holds across the $\frac{1}{2}$-mixtures of the two pairs, i.e. the complementary pairs are mirrored at the same utility level.

The axioms can now be stated. They are continua in α and pose restrictions on preferences over α-mixtures of constant equivalents of acts for duos of complementary pairs.

A.4(α). α-Monotonicity. For $f, f', g, g' \in \mathcal{F}$ and $\alpha \geq \frac{1}{2}$ ($\alpha \leq \frac{1}{2}$), if $(f, f'; g, g')$ is a duo of complementary pairs and $f(s) \succcurlyeq g(s)$ for all $s \in S$, then
$\alpha x_f + (1-\alpha)x_{f'} \succcurlyeq (\preccurlyeq) \alpha x_g + (1-\alpha)x_{g'}$.

A.5(α). α-Ambiguity Attitude. For $f, f', g, g' \in \mathcal{F}$, $\alpha \geq \frac{1}{2}$ ($\alpha \leq \frac{1}{2}$), if $(f, f'; g, g')$ is a duo of complementary pairs, then $\alpha x_f + (1-\alpha)x_{f'} \succcurlyeq (\preccurlyeq) \alpha x_g + (1-\alpha)x_{g'}$ implies $\alpha x_{\beta f+(1-\beta)g} + (1-\alpha)x_{\beta f'+(1-\beta)g'} \succcurlyeq (\preccurlyeq) \alpha x_g + (1-\alpha)x_{g'}$ for all $\beta \in [0,1]$.

The axioms need some "help" to unfold their force. $A.1$, $A.3$ and $A.7$ guarantee the existence of constant equivalents (Lemma 1). Furthermore $A.2$ guarantees that preference is unaffected by the choice of constant equivalents.[5] Thus $A.1$, $A.2$, $A.3$ and $A.7$ are sufficient for the axioms $A.4(\alpha)$ and $A.5(\alpha)$ to unfold their force.[6] Lemma 2 shows that under these axioms, the case $\alpha = 1$ in $A.4(\alpha)$ and $A.5(\alpha)$ is equivalent to $A.4$ and $A.5$, respectively, and the case $\alpha = 0$ is equivalent to $A.4$ and $A.5'$, respectively. Thus these well-known axioms are, under standard conditions, the extreme cases within the introduced continua.

Lemma 2. *Assume* $A.1, A.2, A.3$ *and* $A.7$*, then*

1. $A.4(1) \iff A.4$*;* $A.4(0) \iff A.4$*.*
2. $A.5(1) \iff A.5$*;* $A.5(0) \iff A.5'$*.*

Remark 1. In Lemma 2, $A.2$ can be significantly weakened to the following condition: For $x, y, z \in X$ and $\beta \in (0,1)$, $x \sim y$ if and only if $\beta x + (1-\beta)z \sim \beta y + (1-\beta)z$. The reason is that this condition is sufficient to guarantee the above mentioned independence of choice of constant equivalent, which is all that $A.2$ is needed for here.

Lemma 1 and Lemma 2 directly deliver alternative axiomatizations of 1-MEU and 0-MEU as they imply

$$A.1, A.2, A.3, A.4(1), A.5(1), A.7 \iff A.1, A.2, A.3, A.4, A.5;$$
$$A.1, A.2, A.3, A.4(0), A.5(0), A.7 \iff A.1, A.2, A.3, A.4, A.5'.$$

Theorem 1 shows that this pattern generalizes beyond these extreme cases.

Theorem 1. *Let* $\succcurlyeq$ *be a binary relation on* $\mathcal{F}$*. For each* $\alpha \in [0,1]\backslash\{\frac{1}{2}\}$ *the following are equivalent:*

1. $\succcurlyeq$ *satisfies* $A.1$*,* $A.2$*,* $A.3$*,* $A.4(\alpha)$*,* $A.5(\alpha)$ *and* $A.7$*.*

[5] If x_f and x'_f are both constant equivalents of $f \in \mathcal{F}$, then $A.2$ implies that $\alpha x_f + (1-\alpha)x_g \sim \alpha x'_f + (1-\alpha)x_g$ for all $g \in \mathcal{F}$.

[6] See section 5.2.2 for a discussion on why the axioms need to be stated as restrictions on mixtures of constant equivalents of acts, and not the acts themselves.

2. $\succcurlyeq$ has an α-MEU representation, i.e. there exists an affine $u : X \to \mathbb{R}$ and a nonempty, weak compact and convex set $\mathcal{C} \subseteq \Delta(S, \Sigma)$ such that $\succcurlyeq$ is represented by*

$$V(f) = \alpha \min_{P \in \mathcal{C}} \int u(f) dP + (1 - \alpha) \max_{P \in \mathcal{C}} \int u(f) dP.$$

Furthermore the function u is unique up to positive affine transformations. If $A.6$ is added the set $\mathcal{C}$ is unique.

Remark 2. Theorem 1 relies on a fixed value of α. It is immediate that there is a version of the theorem which does not need this fixation: A binary relation $\succcurlyeq$ satisfies $A.1$, $A.2$, $A.3$, $A.7$ as well as for some $\alpha^* \in [0, 1] \backslash \{\frac{1}{2}\}$ also $A.4(\alpha^*)$ and $A.5(\alpha^*)$ if and only if there exists an $\alpha' \in [0, 1] \backslash \{\frac{1}{2}\}$ such that $\succcurlyeq$ has an α'-MEU representation. Furthermore there exist values α^* and α' which coincide.

This version does not use α as an input but rather relies on an existential quantifier. I consider such a version less compelling as uniqueness of representation is lost and the axiomatic requirements are hard to test.

Remark 3. The uniqueness result in Theorem 1 is not new. It appears in Frick, Iijima, and Yaouanq (2020).

3.2 Idea of the proof of Theorem 1

For the proof of Theorem 1 I use a novel technique which utilizes the existing axiomatization of a special case of the model to axiomatize the general model. In this particular case I make use of the axiomatization of MEU preferences by Gilboa and Schmeidler (1989) to axiomatize the general α-MEU model.

Lemma 5 is the heart of the proof and states the following: Assume $\alpha \neq \frac{1}{2}$ and let $I, I' : B_0(\Sigma) \to \mathbb{R}$ be two functionals[7] such that

$$I'(a) = \frac{\alpha I(a) + (1 - \alpha) I(-a)}{2\alpha - 1} \tag{1}$$

[7] $B_0(\Sigma)$ is the set of real-valued Σ-measurable simple functions, see Section 6.2 for details.

for all $a \in B_0(\Sigma)$. Then I' satisfies certainty independence, homogeneity of degree 1, monotonicity and concavity if and only if I satisfies certainty independence, homogeneity of degree 1, α-monotonicity and α-concavity.[8]

The stated properties of I' are exactly the properties of a functional which induce MEU preferences, as shown in Lemma 3.3 of Gilboa and Schmeidler (1989). Lemma 7 shows that equation (1) is the well-defined relationship between an α-MEU functional I and its "pessimistic MEU twin" I'. This implies that I induces α-MEU preferences if and only if it satisfies certainty independence, homogeneity of degree 1, α-monotonicity and α-concavity. To achieve a characterization of α-MEU, the latter two properties need to be "translated" into axioms, the result of this being $A.4(\alpha)$ and $A.5(\alpha)$.

I thus derive the axioms and prove the representation result for α-MEU by utilizing the existing results for MEU, a special case of the model.[9]

4 Interpretation of $A.4(\alpha)$ and $A.5(\alpha)$ and multiplicity of representation

In this section I shed more light on the introduced axioms and illustrate how the strength of the axioms change in the parameter α. These insights allow a deeper understanding of the phenomenon of multiplicity of representation inherent in the α-MEU model.

4.1 A closer look at $A.4(\alpha)$ and $A.5(\alpha)$

Throughout this subsection I assume $A.1, A.2, A.3$ and $A.7$ which guarantees the existence of constant equivalents as well as the existence of a representation functional $V : \mathcal{F} \to \mathbb{R}$, unique up to positive affine transformations (see section 6.1). Furthermore consider a duo of complementary pairs $(f, f'; g, g')$ and $\alpha \in (\frac{1}{2}, 1)$ (the case $\alpha \in (0, \frac{1}{2})$ being symmetrical and thus omitted or only hinted upon). For the proofs of the claims made in the following, see the proof of Lemma 3 in the appendix.

[8] See Section 6.2 in the appendix for the relevant details on these properties.

[9] Note that this technique cannot be used for α-MEU when $\alpha = \frac{1}{2}$ as the relationship between the $\frac{1}{2}$-MEU functional and its "pessimistic twin" is not well-defined.

Discussion of $A.4(\alpha)$ Assume that f dominates g, i.e. $f(s) \succcurlyeq g(s)$ for all $s \in S$. It is straightforward to see that $A.4(\alpha)$ holds if and only if V satisfies a property that I call *α-monotonicity*:

$$\alpha V(f) + (1-\alpha)V(f') \geq \alpha V(g) + (1-\alpha)V(g'). \tag{2}$$

Since the domination of f over g implies that g' dominates f', it must simultaneously hold that

$$\alpha V(g') + (1-\alpha)V(g) \geq \alpha V(f') + (1-\alpha)V(f). \tag{3}$$

The property α-monotonicity implies standard monotonicity, i.e. the case $\alpha = 1$, which shows that $A.4(\alpha)$ implies $A.4$. Furthermore, for all α in the considered range $(\frac{1}{2}, 1)$, α-monotonicity of V implies

$$f \succ g \iff g' \succ f'. \tag{4}$$

As $A.4$ does not rule out $f \succ f'$ and $g' \sim g$, the symmetry condition in (4) implies that $A.4(\alpha)$ is a real strengthening of $A.4$. But $A.4(\alpha)$ implies more, α-specifically! Rearranging (2) and (3) results in

$$\alpha(V(f) - V(g)) \geq (1-\alpha)(V(g') - V(f')) \tag{5}$$

and

$$\alpha(V(g') - V(f')) \geq (1-\alpha)(V(f) - V(g)), \tag{6}$$

respectively. This shows that when α is close to $\frac{1}{2}$, $V(f) - V(g)$ must be close to $V(g') - V(f')$, otherwise either (5) or (6) is violated. When α is close to 1, there are weaker restrictions on $V(f) - V(g)$ vs. $V(g') - V(f')$. This shows that $A.4(\alpha)$ strictly increases in strength as α approaches $\frac{1}{2}$. The parameter α poses restrictions on the ratio of the gains when going from dominated to dominant acts for duos of complementary pairs.

Discussion of $A.5(\alpha)$ For the purpose of the following discussion I abuse notation by writing $\min\{f, g\}$ for the (weakly) worse act amongst f and g. As shown in Lemma 4 in the appendix, $A.5(\alpha)$ holds if and only if V satisfies a property that I call *α-concavity*:

$$\begin{aligned}&\alpha V(\beta f+(1-\beta)g)+(1-\alpha)V(\beta f'+(1-\beta)g')\\ \geq&\alpha(\beta V(f)+(1-\beta)V(g))+(1-\alpha)(\beta V(f')+(1-\beta)V(g')).\end{aligned} \tag{7}$$

This condition is implied by and therefore weaker than standard concavity, i.e. the case $\alpha=1$, which shows that $A.5$ implies $A.5(\alpha)$. In particular α-concavity of V does not guarantee $\beta f+(1-\beta)g \succcurlyeq \min\{f,g\}$ for all $\beta\in[0,1]$, i.e. preferences may violate $A.5$, thus $A.5(\alpha)$ is indeed strictly weaker. $A.5(\alpha)$ does however imply the weaker condition

$$\beta f+(1-\beta)g \prec \min\{f,g\} \implies \beta f'+(1-\beta)g' \succ \min\{f',g'\}. \tag{8}$$

Thus a violation of preference for mixing amongst acts must be compensated by a preference for mixing amongst complements. But $A.5(\alpha)$ implies more, α-specifically! Rearranging (7) results in

$$\begin{array}{lll}(1-\alpha) & & [V(\beta f'+(1-\beta)g')-\beta V(f')-(1-\beta)V(g')]\\ \geq\ \ \alpha & & [-V(\beta f+(1-\beta)g)+\beta V(f)+(1-\beta)V(g)].\end{array} \tag{9}$$

Assume that $V(\beta f+(1-\beta)g)<\beta V(f)+(1-\beta)V(g)$, i.e. concavity is violated. In order for inequality (9) to be satisfied a necessary requirement is $V(\beta f'+(1-\beta)g')>\beta V(f')+(1-\beta)V(g')$. Thus a violation of concavity must be compensated by concavity amongst complements. How high this compensation must be depends on α: When α is close to 1, compensation is more pronounced then when α is close to $\frac{1}{2}$. This implies that $A.5(\alpha)$ strictly decreases in strength as α approaches $\frac{1}{2}$. The parameter α poses restrictions on concavity/convexity ratios for duos of complementary pairs.

Lemma 3 summarizes the key relationships of the axioms in the parameter α.

Lemma 3. *Assume $A.1, A.2, A.3$ and $A.7$,*[10] *then*

1. *$A.4(\alpha) \implies A.4(\alpha')$ whenever $\frac{1}{2}<\alpha<\alpha'\leq 1$ or $0\leq\alpha'<\alpha<\frac{1}{2}$. The reverse*

[10]Once again, $A.2$ can be relaxed to the much weaker condition from Remark 1.

directions do not hold. Furthermore $A.4(\alpha) \iff A.4(1-\alpha)$ *for all* α.

2. $A.5(\alpha) \implies A.5(\alpha')$ *whenever* $\frac{1}{2} < \alpha' < \alpha \leq 1$ *or* $\frac{1}{2} > \alpha' > \alpha \geq 0$. *The reverse directions do not hold.*

Note how the relationships differ: Whereas $A.4(\alpha)$ increases in strength towards $\frac{1}{2}$, the opposite is the case for $A.5(\alpha)$.[11] A consequence of Lemma 3 thus is that the axiomatic characterization of α-MEU in Theorem 1 is different for each $\alpha \in [0,1]\backslash\{\frac{1}{2}\}$, with none of these axiomatic systems being nested in another.

4.2 Multiplicity of representation

It is well known that a preference relation may have multiple α-MEU representations (Ghirardato, Maccheroni, and Marinacci (2002), Siniscalchi (2006), Frick, Iijima, and Yaouanq (2020)). Proposition 1 in Frick, Iijima, and Yaouanq (2020) sheds significant light on this phenomenon. In particular it states that, when preferences have some α-MEU representation but none with a singleton prior set, the range of α-values that allow a representation is necessarily a subset of one of the mutually exclusive sets $[0, \frac{1}{2})$, $(\frac{1}{2}, 1]$ and $\{\frac{1}{2}\}$. Furthermore their proposition provides the exact relationships of different α-MEU representations. In the first two cases, the prior sets are γ-*expansions* of each other (see their equation (4)).

Theorem 1 and Lemma 3 complement the existing insights by tracing back the phenomenon to preference conditions, allowing a deeper understanding of the multiplicity phenomenon. For a fixed α, Theorem 1 lays open the preference conditions equivalent to the existence of an α-MEU representation and does not rule out additional representations with other α-values. Consider now a preference relation which has an α-MEU representation for some $\alpha \in (\frac{1}{2}, 1]$. There exists a smallest α-value such that $A.4(\alpha)$ is satisfied and a largest α-value such that $A.5(\alpha)$ is satisfied (see the proof of Corollary 1 in the appendix). The relationships unveiled in Lemma 3 imply that the α-values between these two extremes are exactly those that allow an α-MEU representation. The following corollary states that this interval exists in general.

[11] In Section 5.3 I illustrate that this logic continues to hold when $\alpha = \frac{1}{2}$ is also considered.

Corollary 1. *For any preference relation with some α-MEU representation, the set of α-values that allow an α-MEU representation is a closed interval.*

Of particular interest are the extreme cases of this interval. Considering the case where all representations are within $[0, \frac{1}{2})$ or $(\frac{1}{2}, 1]$, the bound of the interval furthest from $\frac{1}{2}$ induces the representation with the smallest prior set. The bound closest to $\frac{1}{2}$ induces the representation with the largest prior set. The following proposition shows that this largest prior set always touches the boundary of the simplex, i.e. it contains a Σ-measurable element P such that $P(E) = 0$ for some $\emptyset \neq E \in \Sigma$.

Proposition 1. *For any preference relation with some α-MEU representation with $\alpha \neq \frac{1}{2}$, the representation with the largest prior set intersects the boundary of the simplex $\Delta(S, \Sigma)$.*

Proposition 1 implies in particular that when representation is unique, i.e. the interval from Corollary 1 is a singleton, the resulting prior set touches the boundary of the simplex $\Delta(S, \Sigma)$.

The following example provides intuition of these results.

Example 1. Consider the state space $S = \{s_1, s_2, s_3\}$, assume for simplicity that acts map to utilities and consider the distribution $P = (\frac{3}{8}, \frac{1}{4}, \frac{3}{8})$ over S. Consider the preference relation $\succcurlyeq$ represented by $\alpha^* = \frac{3}{4}$ and $\mathcal{C}^* = B_{\frac{1}{8}}(P)$.[12] Then $\underline{\alpha} = \frac{5}{8}, \overline{\mathcal{C}} = B_{\frac{1}{4}}(P)$ corresponds to the representation of $\succcurlyeq$ with the smallest α-value and the largest prior set. Furthermore $\overline{\mathcal{C}}$ intersects the boundary of the simplex as $(\frac{1}{2}, 0, \frac{1}{2}) \in \overline{\mathcal{C}}$ and $\frac{5}{8}$ is the smallest α-value such that $A.4(\alpha)$ is satisfied. For $\alpha \in (\frac{1}{2}, \frac{5}{8})$, $A.4(\alpha)$ is violated[13] and the resulting "prior set" is not a subset of the simplex. $\overline{\alpha} = 1, \underline{\mathcal{C}} = B_{\frac{1}{16}}(P)$ correspond to the representation of $\succcurlyeq$ with the largest α-value and the smallest prior set. Figure 1 illustrates this.

[12] $B_a(P)$ is the ball around P with radius $a \geq 0$.

[13] To see this, consider the acts $f = (1, 0, 1)$, $f' = (-1, 0, -1)$ and $g = g' = (0, 0, 0)$. Clearly, $(f, f'; g, g')$ is a duo of complementary pairs and f dominates g. Simple calculations show that the evaluations of these acts are $V(f) = \frac{3}{16}$, $V(f') = -\frac{5}{16}$ and $V(g) = V(g') = 0$. For $\alpha \in (\frac{1}{2}, \frac{5}{8})$ we have $\alpha V(f) + (1-\alpha)V(f') < \alpha V(g) + (1-\alpha)V(g')$, a violation of $A.4(\alpha)$.

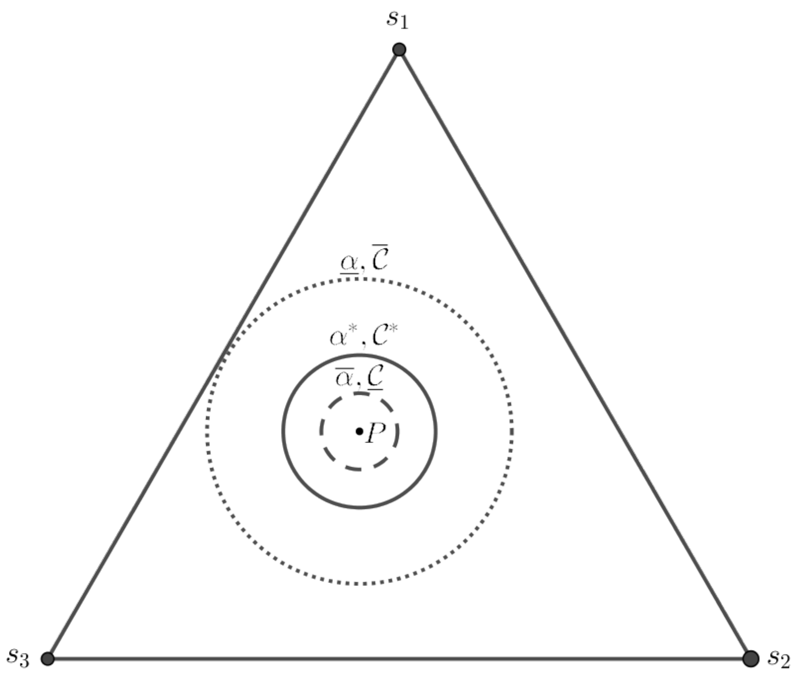

Figure 1: Illustration of the representation $\alpha^*, \mathcal{C}^*$ as well as the extreme alternative representations.

5 Discussion

This section discusses related literature. Furthermore I briefly examine the role of mixtures of constant equivalents of acts, on which the introduced axioms rely. I proceed with a discussion on the case $\alpha = \frac{1}{2}$. The section ends with a conclusion.

5.1 Related literature

α-MEU There is a large theoretical literature on the α-MEU model.

Ghirardato, Maccheroni, and Marinacci (2004) achieve a representation of invariant biseparable preferences, i.e. preferences which satisfy $A.1 - A.4$, a more general class than α-MEU.[14] Their representation functional is characterized by a prior set which coincides with the Bewley set (Bewley (2002)) of $\succcurlyeq^*$, the largest subrelation satisfying independence, as well as an act-dependent pessimism index. Their contribution showed that axiomatics can go beyond both uncertainty aversion and the Choquet expected utility model of

[14]See Amarante (2009) and Chandrasekher, Frick, Iijima, and Le Yaouanq (2021) for alternative representations of this class of preferences.

Schmeidler (1989). They also show that if an additional axiom is added, the pessimism index becomes act-independent, i.e. a characterization of α-MEU preferences is achieved in which the prior set equals the Bewley set. This axiom relies on the subrelation $\succcurlyeq^*$ as well as the concept of "possible" certainty equivalents of acts. Eichberger, Grant, Kelsey, and Koshevoy (2011) show that when the state space is finite, the approach of Ghirardato, Maccheroni, and Marinacci (2004) implies $\alpha \in \{0, 1\}$. Chateauneuf, Eichberger, and Grant (2007) characterize preferences that can be represented by a Choquet integral (Choquet (1954)) with respect to a *neo-additive capacity.* This class of preferences can be represented by an α-MEU functional with respect to a prior set of the form $\mathcal{C} = (1-\epsilon)\{P\} + \epsilon\Delta(S, \Sigma)$, where $\epsilon \in [0, 1]$ and $P \in \Delta(S, \Sigma)$. Such prior sets are shrunk versions of the simplex. Gul and Pesendorfer (2015) achieve an α-MEU representation in which $\mathcal{C}$ corresponds to the set of priors that agree with a prior P on some σ-algebra. Preferences reduce to subjective expected utility when restricting attention to acts which are measurable with respect to this σ-algebra (*ideal acts*). The interpretation is that the decision maker is completely ignorant beyond the bounds implied by P. Frick, Iijima, and Yaouanq (2020) achieve a characterization of α-MEU by incorporating the concept of objective rationality in the style of Gilboa, Maccheroni, Marinacci, and Schmeidler (2010), where a subrelation $\succcurlyeq^*$ reflects the possibly incomplete ranking over acts that the decision maker considers uncontroversial. Kopylov (2003) derives the prior set of the model from a particular class of subjectively risky acts. Klibanoff, Mukerji, Seo, and Stanca (2021) provide a characterization of α-MEU in which the state space has an infinite product structure and additional symmetry conditions are imposed. Jaffray (1994) represents uncertainty by belief functions and characterizes the model for a given set of priors.

There is a strand of literature that models imprecise information about the objective lotteries in the second stage of the Anscombe-Aumann framework by considering *bi-lotteries*, the set of convex combinations of two lotteries. Hill (2019a) provides a characterization in such an enriched setup. In his characterization result, preferences over bi-lotteries are also of the α-MEU type, hereby building on Olszewski (2007) and Vierø (2009).

$A.4(\alpha)$ **and** $A.5(\alpha)$ I am not aware of other literature in which axioms are considered that restrict preferences over mixtures amongst constant equivalents. See also the discussion in Section 5.2.2. Whereas complementary pairs of acts are already introduced and utilized in an axiom in Siniscalchi (2009) as well as Hill (2019a), the concept of stating an axiom for duos of complementary pairs is novel.

As shown in Lemma 3, under weak assumptions, $A.4(\alpha)$ is a strengthening of $A.4$. Nascimento and Riella (2010) consider different strengthenings of $A.4$ to achieve alternative axiomatizations of MEU and variational preferences (Maccheroni, Marinacci, and Rustichini (2006)). The axiom $P3$ from Savage (1954) is a strengthening of $A.4$ but corresponds merely to a strict version of it. There is a more significant literature which considers weakenings of $A.4$. Axiom $A.7$, being a weakening of $A.4$, is not new. Nascimento and Riella (2010) mention it and point out its role on the existence of constant equivalents. Blavatskyy (2007) mentions it in the context of risk. Chambers and Echenique (2016) discuss it under an equivalent form and the name *uniform monotonicity.* Bullen (2013) as well as Merikoski, Halmetoja, and Tossavainen (2009) rely on $A.7$ in the axiomatization of means. The condition also appears in the book Aczél (1966) on functional equations (p. 342). Other weakenings of $A.4$ have been discussed. Hill (2019b) considers *state consistency*, an axiom that he shows to be crucial in the context of state-dependent utility. Grant and Polak (2013) consider the axiom *Substitution* which they show to be sufficient for state independence.

As shown in Lemma 3, under weak conditions, $A.5(\alpha)$ is a weakening of $A.5$. The idea to achieve new representation results by weakening $A.5$ in the axiomatization of MEU by Gilboa and Schmeidler (1989) has been considered before. As already mentioned, Ghirardato, Maccheroni, and Marinacci (2004), Amarante (2009) and Chandrasekher, Frick, Iijima, and Le Yaouanq (2021) drop $A.5$ altogether and achieve representation results for the resulting class of invariant biseparable preferences. Hartmann and Kauffeldt (2019) introduce a *Hierarchy of Ambiguity Aversion* in which higher levels of ambiguity aversion are characterized by a more pronounced preference for mixing amongst acts. The highest level corresponds to $A.5$, the lowest level corresponds to the axiom *preference* *for complete hedges* as introduced in Grant and Polak (2013). Chateauneuf and Tallon (2002) use the name *sure diversification* for this axiom and, in the Choquet expected utility model, show that it characterizes *balanced capacities*, i.e. set functions with a non-empty core. Chandrasekher, Frick, Iijima, and Le Yaouanq (2021) relax $A.5$ by also introducing a hierarchy. Their strongest level corresponds to preference for complete hedges in Grant and Polak (2013).

5.2 Axioms on mixtures of constant equivalents of acts: A closer look

The axioms introduced in this paper restrict preferences over mixtures of constant equivalents of acts. In this subsection I illustrate that the axioms have an ex-ante mixing interpretation which sidesteps the construction of constant equivalents.[15] Furthermore I show that the axioms are meaningful as stated, i.e. they cannot be changed to statements on the acts themselves instead of their constant equivalents.

5.2.1 Ex-ante mixing

Let $\oplus$ denote ex-ante mixing: For two acts $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, $\alpha f \oplus (1-\alpha)g$ is the α-mixture of f and g before the state is realized (see Ke and Zhang (2020), Nascimento and Riella (2011), Saito (2015), Seo (2009) as well as Anscombe and Aumann (1963) for details on ex-ante vs. ex-post mixing).

Consider the following weak conditions:

a) For $x, y \in X$ and $\alpha \in [0,1]$,

$$\alpha x + (1-\alpha)y \sim \alpha x \oplus (1-\alpha)y.$$

b) For $f, f' \in \mathcal{F}$ with (f, f') being a complementary pair and $\alpha \in [0,1]$,

$$\alpha f \oplus (1-\alpha)f' \sim \alpha x_f \oplus (1-\alpha)x_{f'}.$$

[15] I am grateful to Ryota Iijima for making me aware of this.

Condition a) is Anscombe and Aumann's *reversal of order* axiom restricted to constant acts. Condition b) states, for complementary pairs, indifference between the ex-ante mixture of the acts and the ex-ante mixture of their constant equivalents.

The following axioms do not rely on preferences over constant equivalents but rather on ex-ante mixing of the acts themselves.

Ex-ante *A.4(α).* α*-Monotonicity.* For $f, f', g, g' \in \mathcal{F}$ and $\alpha \geq \frac{1}{2}$ ($\alpha \leq \frac{1}{2}$), if $(f, f'; g, g')$ is a duo of complementary pairs and $f(s) \succcurlyeq g(s)$ for all $s \in S$, then $\alpha f \oplus (1-\alpha) f' \succcurlyeq (\preccurlyeq) \alpha g \oplus (1-\alpha) g'$.

Ex-ante *A.5(α).* α*-Ambiguity Attitude.* For $f, f', g, g' \in \mathcal{F}$, $\alpha \geq \frac{1}{2}$ ($\alpha \leq \frac{1}{2}$), if $(f, f'; g, g')$ is a duo of complementary pairs, then $\alpha f \oplus (1-\alpha) f' \succcurlyeq (\preccurlyeq) \alpha g \oplus (1-\alpha) g'$ implies $\alpha(\beta f + (1-\beta) g) \oplus (1-\alpha)(\beta f' + (1-\beta) g') \succcurlyeq (\preccurlyeq) \alpha g \oplus (1-\alpha) g'$ for all $\beta \in [0,1]$.

It is immediate that under a) and b), $A.4(\alpha)$ and $A.5(\alpha)$ are equivalent to Ex-ante $A.4(\alpha)$ and Ex-ante $A.5(\alpha)$, respectively.[16] This shows that axioms which state preference conditions over mixtures of constant equivalents of acts, in particular $A.4(\alpha)$ and $A.5(\alpha)$, have an ex-ante mixing interpretation, allowing a basic intuition that sidesteps the procedure of constructing constant equivalents.

5.2.2 Meaningfulness of mixtures of constant equivalents of acts

The reader may ask why the introduced axioms rely on constant equivalents. Why can't we consider the versions of $A.4(\alpha)$ and $A.5(\alpha)$ which consider the acts themselves?

In general,

$$\alpha f + (1-\alpha) g \sim \alpha x_f + (1-\alpha) x_g \tag{10}$$

does not hold for all $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, thus it potentially makes a difference whether mixtures of acts or mixtures of their constant equivalents are considered.

The following proposition shows that condition (10) is equivalent to the standard independence axiom under weak conditions.

A.8. Independence For $f, g, h \in \mathcal{F}$, $\alpha \in (0,1)$, $f \succcurlyeq g \iff \alpha f + (1-\alpha) h \succcurlyeq \alpha g + (1-\alpha) h$.

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[16] $\alpha x_f + (1-\alpha) x_{f'} \overset{a)}{\sim} \alpha x_f \oplus (1-\alpha) x_{f'} \overset{b)}{\sim} \alpha f + \oplus (1-\alpha) f'$.

Proposition 2. *Let $\succcurlyeq$ be a binary relation on $\mathcal{F}$ which satisfies $A.1$ and assume that constant equivalents exist. Then $A.8$ implies (10) for all acts and mixtures. Under risk independence (i.e. independence restricted to the set of constant acts X), the reverse is also true.*

Proposition 2 shows that whenever standard independence is violated it may be meaningful whether we consider axioms on mixtures of acts or axioms on mixtures of their constant equivalents. If for instance in Theorem 1, $A.4(\alpha)$ and $A.5(\alpha)$ were replaced by the versions where the acts themselves are considered, the properties of the representation functional would change. Indeed it is easily seen that the resulting axioms would induce 1-MEU preferences for $\alpha > \frac{1}{2}$ and 0-MEU preferences for $\alpha < \frac{1}{2}$.

Thus in light of Proposition 2 and the axiomatization of subjective expected utility by Anscombe and Aumann (1963), in which $A.8$ is the crucial axiom, such reasoning applies to all models that are weakenings of SEU and are characterized by axioms on mixing preferences. Replacing those axioms by mixtures of constant equivalents potentially changes the properties of the representation functional and therefore the model that it characterizes. Whether such an approach leads to (other) interesting insights is a question for future research.

5.3 The case $\alpha = \frac{1}{2}$

I do not characterize preferences that have a $\frac{1}{2}$-MEU representation.[17] The reader may ask why one cannot simply use $A.4(\frac{1}{2})$ and $A.5(\frac{1}{2})$ as defined above, i.e.

A.4($\frac{1}{2}$). $\frac{1}{2}$-Monotonicity. For $f, f', g, g' \in \mathcal{F}$, if $(f, f'; g, g')$ is a duo of complementary pairs and $f(s) \succcurlyeq g(s)$ for all $s \in S$, then $\frac{1}{2}x_f + \frac{1}{2}x_{f'} \sim \frac{1}{2}x_g + \frac{1}{2}x_{g'}$.

A.5($\frac{1}{2}$). $\frac{1}{2}$-Ambiguity Attitude. For $f, f', g, g' \in \mathcal{F}$, if $(f, f'; g, g')$ is a duo of complementary pairs, then $\frac{1}{2}x_f + \frac{1}{2}x_{f'} \succcurlyeq \frac{1}{2}x_g + \frac{1}{2}x_{g'}$ implies $\frac{1}{2}x_f + \frac{1}{2}x_{f'} \succcurlyeq \frac{1}{2}x_{\beta f+(1-\beta)g} + \frac{1}{2}x_{\beta f'+(1-\beta)g'} \succcurlyeq \frac{1}{2}x_g + \frac{1}{2}x_{g'}$ for all $\beta \in [0, 1]$.

The logic of Lemma 3 stays valid: Assume $A.1, A.2, A.3$ and $A.7$, then

[17] As already illustrated in Jaffray and Philippe (1997), Ghirardato, Klibanoff, and Marinacci (1998) and Frick, Iijima, and Yaouanq (2020) the case is special.

- For all $\alpha \in [0,1]$ and under $A.4$, $A.4(\frac{1}{2}) \implies A.4(\alpha)$.

- For all $\alpha \in (\frac{1}{2},1]$, $A.5(\alpha) \implies$ the second "$\succcurlyeq$" of $A.5(\frac{1}{2})$.

- For all $\alpha \in [0,\frac{1}{2})$, $A.5(\alpha) \implies$ the first "$\succcurlyeq$" of $A.5(\frac{1}{2})$.

Furthermore, $A.4(\frac{1}{2})$ and $A.5(\frac{1}{2})$ are necessary for a preference relation to have a $\frac{1}{2}$-MEU representation. To see this consider the following axiom.

A.9. Complementary Pair Indifference. For $f, f' \in \mathcal{F}$ with (f, f') being a complementary pair, $\frac{1}{2}x_f + \frac{1}{2}x_{f'} \sim \frac{1}{2}f + \frac{1}{2}f'$.

Under $A.1$, $A.3$ and $A.7$, both $A.4(\frac{1}{2})$ and $A.5(\frac{1}{2})$ are implied by $A.9$[18] which in turn is necessary for a preference relation to have a $\frac{1}{2}$-MEU representation.[19]

The bad news is that $A.1, A.2, A.3, A.4(\frac{1}{2}), A.5(\frac{1}{2})$ and $A.7$ are not sufficient to guarantee a $\frac{1}{2}$-MEU representation. To prove this claim consider the state space $S = \{s_1, s_2, s_3\}$ and the preference functional which assigns to each act its (weakly) second-best consequence. These preferences satisfy $A.1, A.2, A.3, A.7$ and $A.9$ (and thus $A.4(\frac{1}{2})$ and $A.5(\frac{1}{2})$), but do not have a $\frac{1}{2}$-MEU representation.[20]

Thus $A.4(\frac{1}{2})$ and $A.5(\frac{1}{2})$ are not the whole story. I view this as an important question for future research.

5.4 Conclusion

This paper axiomatically characterizes α-MEU preferences in the Anscombe-Aumann framework for all α-values other than $\frac{1}{2}$. The multiplicity of representation inherent in the α-MEU model is traced back to preference conditions.

The introduced continua of axioms $A.4(\alpha)$ and $A.5(\alpha)$ are of a novel type and build on the concept of duos of complementary pairs. The idea of the proof is to build a bridge between the properties of the α-MEU and MEU functionals, thus utilizing the existing characterization for MEU preferences by Gilboa and Schmeidler (1989).

[18] This follows from $A.7$. It implies that for a duo of complementary pairs $(f, f'; g, g')$ we have $\frac{1}{2}f + \frac{1}{2}f' \sim \frac{1}{2}g + \frac{1}{2}g' \sim \frac{1}{2}(\beta f + (1-\beta g) + \frac{1}{2}(\beta f' + (1-\beta)g'))$ for all $\beta \in [0,1]$.

[19] This follows from simple calculations and Lemma 6 in the appendix.

[20] They do have a $\frac{1}{2}$-MEU representation if the requirement that the prior set is a subset of $\Delta(S,\Sigma)$ is dropped. Define $\mathcal{C} = Conv\{(1,1,-1),(1,-1,1),(-1,1,1)\}$, where $(1,1,-1)$ denotes the "distribution" which puts weight 1 on s_1 and s_2 as well as weight -1 on s_3.

6 Appendix

6.1 Proof of Lemma 1, Lemma 2 and Lemma 3

Proof of Lemma 1. Let $\succcurlyeq$ be a binary relation on $\mathcal{F}$ and consider some $f \in \mathcal{F}$. Consider $\overline{x} \in \{f(s')|f(s') \succcurlyeq f(s)\ \forall s \in S\}$ and $\underline{x} \in \{f(s')|f(s) \succcurlyeq f(s')\ \forall s \in S\}$. $A.7$ implies that $\overline{x} \succcurlyeq f \succcurlyeq \underline{x}$. If $\overline{x} \sim f$ or $f \sim \underline{x}$ we are finished, therefore assume $\overline{x} \succ f \succ \underline{x}$. For $\beta \in [0,1]$ denote by $x_\beta = \beta\overline{x} + (1-\beta)\underline{x}$ the β-mix of $\overline{x}$ and $\underline{x}$.

Define $U = \{\beta \in (0,1)|x_\beta \succ f\}$ and $L = \{\gamma \in (0,1)|f \succ x_\gamma\}$. $A.3$ implies that U and L are non-empty. $A.7$ implies

$$\beta > \gamma \quad \forall \beta \in U, \gamma \in L. \tag{11}$$

Define $\underline{\beta} = \inf\limits_{\beta \in U} \beta$ and $\bar{\gamma} = \sup\limits_{\gamma \in L} \gamma$. It follows from (11) that $\underline{\beta} \geq \bar{\gamma}$. Non-emptyness of U and L imply that $1 > \underline{\beta} \geq \bar{\gamma} > 0$. We have three cases:

1. $x_{\underline{\beta}} \sim f$. Then we are finished.

2. $\underline{\beta} \in U$. It follows that $x_{\underline{\beta}} \succ f$. $A.3$ implies that there exists a $\lambda \in (0,1)$ such that $x_{\lambda\underline{\beta}} = \lambda x_{\underline{\beta}} + (1-\lambda)x_2 \succ f$, thus $\lambda\underline{\beta} \in U$ and $\lambda\underline{\beta} < \underline{\beta}$. This contradicts $\underline{\beta} = \inf\limits_{\beta \in U} \beta$.

3. $\underline{\beta} \notin U$ and $x_{\underline{\beta}} \nsim f$. $A.1$ implies $f \succ x_{\underline{\beta}}$. Therefore $\underline{\beta} \in L$. We have $\underline{\beta} \geq \bar{\gamma} = \sup\limits_{\gamma \in L} \gamma \geq \underline{\beta}$, which implies $\underline{\beta} = \bar{\gamma}$. $A.3$ implies the existence of some $\lambda \in (0,1)$ such that $f \succ \lambda x_1 + (1-\lambda)x_{\bar{\gamma}} = x_{\lambda+(1-\lambda)\bar{\gamma}}$, thus $\lambda + (1-\lambda)\bar{\gamma} \in L$ and $\bar{\gamma} < \lambda + (1-\lambda)\bar{\gamma}$. This contradicts $\bar{\gamma} = \sup\limits_{\gamma \in L} \gamma$.

$\square$

Proof of Lemma 2. 1. For $A.4(1) \iff A.4$, the implication " $\Longleftarrow$ " is obvious. For " $\Longrightarrow$ " consider $f, g \in \mathcal{F}$ such that f dominates g. Let $\overline{x}_f$ be the best consequence of f and $\underline{x}_g$ the worst consequence of g. If those are not unique take arbitrary ones. $A.7$ implies $\overline{x}_f \succcurlyeq f \succcurlyeq \underline{x}_g$ as well as $\overline{x}_f \succcurlyeq g \succcurlyeq \underline{x}_g$. Due to $A.3$ there exist for all $s \in S$ some unique $\beta_s^f, \beta_s^g \in [0,1]$ such that $f(s) \sim \beta_s^f \overline{x}_f + (1-\beta_s^f)\underline{x}_g$ and $g(s) \sim \beta_s^g \overline{x}_f + (1-\beta_s^g)\underline{x}_g$.

Define f' and g' by

$$f'(s) = (1 - \beta_s^f)\overline{x}_f + \beta_s^f \underline{x}_g$$
$$g'(s) = (1 - \beta_s^g)\overline{x}_f + \beta_s^g \underline{x}_g$$

for all $s \in S$. By construction $(f, f'; g, g')$ is a duo of complementary pairs. Thus $A.4(1)$ implies $f \succcurlyeq g$. The proof for $A.4(0) \iff A.4$ is analogous.

2. For $A.5(1) \iff A.5$, the " $\Longleftarrow$ " implication is obvious. For " $\Longrightarrow$ ", consider $f, g \in \mathcal{F}$ with $f \succcurlyeq g$. For an arbitrary $\beta \in [0,1]$ we must show $\beta f + (1-\beta)g \succcurlyeq g$. Let $\overline{x}_{f,g}$ and $\underline{x}_{f,g}$ be the best and worst consequence that either f and g result in. If those are not unique take arbitrary ones. Due to $A.3$ and $A.7$ there exist for all $s \in S$ some unique $\beta_s^f, \beta_s^g \in [0,1]$ such that $f(s) \sim \beta_s^f \overline{x}_{f,g} + (1-\beta_s^f)\underline{x}_{f,g}$ and $g(s) \sim \beta_s^g \overline{x}_{f,g} + (1-\beta_s^g)\underline{x}_{f,g}$. Define f' and g' by

$$f'(s) = (1 - \beta_s^f)\overline{x}_{f,g} + \beta_s^f \underline{x}_{f,g}$$
$$g'(s) = (1 - \beta_s^g)\overline{x}_{f,g} + \beta_s^g \underline{x}_{f,g}$$

for all $s \in S$. By construction $(f, f'; g, g')$ is a duo of complementary pairs. Thus $A.5(1)$ implies $\beta f + (1-\beta)g \succcurlyeq g$ for all $\beta \in [0,1]$.

The proof for $A.5(0) \iff A.5'$ is analogous.

$\square$

Before proving Lemma 3 note that a binary relation satisfying $A.1, A.2, A.3$ and $A.7$ can be represented by a representation functional $V : \mathcal{F} \to \mathbb{R}$, unique up to positive affine transformations: $A.1$ and $A.3$ guarantee the existence of an (ordinal) utility function $u : X \to \mathbb{R}$, $A.2$ makes this function affine. Now define $V(f) = u(x_f)$ for any $f \in \mathcal{F}$, where the constant equivalents exist due to Lemma 1. $A.2$ guarantees uniqueness of V up to positive affine transformations. Once this representation functional is established, it is easily seen what properties $A.4(\alpha)$ and $A.5(\alpha)$ imply for it. Let $(f, f'; g, g')$ be a duo of complementary pairs. $A.4(\alpha)$ states that when f dominates g, $\alpha V(f) + (1-\alpha)V(f') \geq$

$\alpha V(g) + (1-\alpha)V(g')$ and $A.5(\alpha)$ states for all $\beta \in [0,1]$ that $\alpha V(f) + (1-\alpha)V(f') \geq \alpha V(g) + (1-\alpha)V(g')$ implies $\alpha V(\beta f + (1-\beta)g) + (1-\alpha)V(\beta f' + (1-\beta)g') \geq \alpha V(g) + (1-\alpha)V(g')$.

Proof of Lemma 3. Throughout the proof let $(f, f'; g, g')$ be a duo of complementary pairs and let $V : \mathcal{F} \to \mathbb{R}$ be some representation functional.

1. Assume that $\frac{1}{2} < \alpha < \alpha' \leq 1$. We need to show that $A.4(\alpha)$ implies $A.4(\alpha')$. First consider the case $\alpha' = 1$. Assume that f dominates g. Note that this implies that g' dominates f'. $A.4(\alpha)$ thus implies not only $\alpha V(f) + (1-\alpha)V(f') \geq \alpha V(g) + (1-\alpha)V(g')$, but also $\alpha V(g') + (1-\alpha)V(g) \geq \alpha V(f') + (1-\alpha)V(f)$. Rearrangements imply

$$\begin{aligned} V(f) - V(g) &\geq \frac{1-\alpha}{\alpha}(V(g') - V(f')) \\ &\geq \underbrace{\frac{1-\alpha}{\alpha}\frac{1-\alpha}{\alpha}}_{<1}(V(f) - V(g)). \end{aligned}$$

This implies $V(f) \geq V(g)$ and thus $A.4(1)$. Now for the general case

$$\begin{aligned} &\alpha'(V(f) - V(g)) + (1-\alpha')(V(f') - V(g')) \\ =&\underbrace{\alpha(V(f) - V(g)) + (1-\alpha)(V(f') - V(g'))}_{\geq 0} \\ +&\underbrace{(\alpha' - \alpha)}_{\geq 0}(\underbrace{V(f) - V(g)}_{\geq 0}\underbrace{-V(f') + V(g')}_{\geq 0}) \\ \geq& 0. \end{aligned}$$

The case $0 \leq \alpha' < \alpha < \frac{1}{2}$ is analogous.

To show that $A.4(\alpha) \iff A.4(1-\alpha)$ assume that f dominates g. Again this implies that g' dominates f'. Without loss of generality assume $\alpha > \frac{1}{2}$ and that $A.4(\alpha)$ holds, i.e.

$$\alpha V(f) + (1-\alpha)V(f') \succcurlyeq \alpha V(g) + (1-\alpha)V(g'), \tag{12}$$

$$\alpha V(g') + (1-\alpha)V(g) \succcurlyeq \alpha V(f') + (1-\alpha)V(f). \tag{13}$$

Note that $1-\alpha < \frac{1}{2}$ thus for $A.4(1-\alpha)$ to hold we must have

$$(1-\alpha)V(f) + \alpha V(f') \preccurlyeq (1-\alpha)V(g) + \alpha V(g'), \tag{14}$$

$$(1-\alpha)V(g') + \alpha V(g) \preccurlyeq (1-\alpha)V(f') + \alpha V(f). \tag{15}$$

But (14) corresponds to (13) and (15) corresponds to (12). Thus $A.4(1-\alpha)$ holds.

2. We prove that $A.5(\alpha) \implies A.5(\alpha')$ whenever $\frac{1}{2} < \alpha' < \alpha \leq 1$. Consider $\beta \in [0,1]$. For better readability define $v_1 = V(\beta f + (1-\beta)g)$, $v_1' = V(\beta f' + (1-\beta)g')$, $v_2 = \beta V(f) + (1-\beta)V(g)$ and $v_2' = \beta V(f') + (1-\beta)V(g')$.

Since we assume that $A.5(\alpha)$ holds we have

$$\alpha v_1 + (1-\alpha)v_1' \geq \alpha v_2 + (1-\alpha)v_2', \tag{16}$$

$$\alpha v_1' + (1-\alpha)v_1 \geq \alpha v_2' + (1-\alpha)v_2. \tag{17}$$

We must show

$$\alpha' v_1 + (1-\alpha')v_1' \geq \alpha' v_2 + (1-\alpha')v_2'. \tag{18}$$

Rearranging (16) and (17) gives $v_1 - v_2 \geq \frac{(1-\alpha)(v_2'-v_1')}{\alpha}$ and $v_1' - v_2' \geq \frac{(1-\alpha)(v_2-v_1)}{\alpha}$, respectively. In addition either $v_1 \geq v_2$ or $v_1' \geq v_2'$ (or both) must hold, since when both inequalities fail, $A.5(\alpha)$ is violated. First assume that $v_1 \geq v_2$. Rearranging (18) and plugging in above gives

$$\begin{aligned}\alpha'(v_1 - v_2) + (1-\alpha')(v_1' - v_2') &\geq \alpha'(v_1 - v_2) + (1-\alpha')\frac{(1-\alpha)(v_2 - v_1)}{\alpha} \\ &= (v_1 - v_2)\frac{\alpha' + \alpha - 1}{\alpha} \geq 0,\end{aligned}$$

where the last inequality follows from $\frac{1}{2} < \alpha' < \alpha$. Now assume $v_1' \geq v_2'$. Then

$$\begin{aligned}\alpha'(v_1 - v_2) + (1-\alpha')(v_1' - v_2') &\geq \alpha'\frac{1-\alpha}{\alpha}(v_2' - v_1') + (1-\alpha')(v_1' - v_2') \\ &= (v_1' - v_2')\frac{\alpha - \alpha'}{\alpha} \geq 0,\end{aligned}$$

where the last inequality follows from $\alpha' < \alpha$. The case $A.5(\alpha) \implies A.5(\alpha')$ whenever

$\frac{1}{2} > \alpha' > \alpha \geq 0$ is analogous.

$\square$

6.2 Proof of Theorem 1

Denote by $B_0(\Sigma)$ the set of real-valued Σ-measurable simple functions. If $f \in \mathcal{F}$ and $u : X \to \mathbb{R}$, $u(f)$ is the element of $B_0(\Sigma)$ defined by $u(f)(s) = u(f(s))$ for all $s \in S$. For $c \in \mathbb{R}$, let $c^* \in B_0(\Sigma)$ be the constant function taking value c. With abuse of notation the set of constant functions in $B_0(\Sigma)$ is referred to as $\mathbb{R}$. For $a \in B_0(\Sigma)$ define $-a \in B_0(\Sigma)$ by $(-a)(s) = -(a(s))$ for all $s \in S$. Note that for $a, b \in B_0(\Sigma)$ we have $\frac{1}{2}a + \frac{1}{2}(-a) = 0^*$ and for $\beta \in [0, 1]$, $\beta(-a) + (1-\beta)(-b) = -(\beta a + (1-\beta)b)$.

Consider a functional $I : B_0(\Sigma) \to \mathbb{R}$. I introduce novel properties of functionals which provide generalizations of the concepts of monotonicity as well as concavity/convexity. For $\alpha > \frac{1}{2}$ ($\alpha < \frac{1}{2}$) I call a functional *α-monotonic* if for all $a, b \in B_0(\Sigma)$ with $a \geq b$, i.e. $a(s) \geq b(s)$ for all $s \in S$,

$$\alpha I(a) + (1-\alpha)I(-a) \geq (\leq) \alpha I(b) + (1-\alpha)I(-b). \tag{19}$$

The property 1-monotonic corresponds to standard monotonicity. For $\alpha > \frac{1}{2}$ ($\alpha < \frac{1}{2}$) I call a functional *α-concave* if for all $a, b \in B_0(\Sigma)$ and $\beta \in [0, 1]$

$$\begin{aligned} &\alpha I(\beta a + (1-\beta)b) + (1-\alpha)I(\beta(-a) + (1-\beta)(-b)) \\ &\geq (\leq) \alpha(\beta I(a) + (1-\beta)I(-a)) + (1-\alpha)(\beta I(b) + (1-\beta)I(-b)). \end{aligned} \tag{20}$$

The property 1-concavity corresponds to standard concavity and 0-concavity to convexity. The following properties of functionals are well-known, see for instance Gilboa and Schmeidler (1989). A functional is called *C-independent* if $I(a + c^*) = I(a) + I(c^*)$ for all $a \in B_0(\Sigma)$ and $c^* \in \mathbb{R}$. It is called *homogenous of degree 1* if $I(\gamma a) = \gamma I(a)$ for all $a \in B_0(\Sigma)$ and $\gamma \geq 0$.

The set of bounded, finitely additive set functions on Σ is denoted by $ba(\Sigma)$. The set of probabilities on Σ is denoted by $pc(\Sigma)$. A functional $I : B_0(\Sigma) \to \mathbb{R}$ has an *α-MEU* *representation* if there exists a nonempty, weak* closed and convex set $\mathcal{C} \subseteq pc(\Sigma)$, called a *prior set*, and $\alpha \in [0,1]$ such that for all $a \in B_0(\Sigma)$

$$I(a) = \alpha \min_{P \in \mathcal{C}} \int a dP + (1-\alpha) \max_{P \in \mathcal{C}} \int a dP. \tag{21}$$

The following lemma exactly pins down the properties of the representation functional of a preference relation satisfying the axioms of Theorem 1.

Lemma 4. *Let $\succcurlyeq$ be a binary relation over $\mathcal{F}$ and let $\alpha \in [0,1] \backslash \{\frac{1}{2}\}$. The following are equivalent:*

1. *$\succcurlyeq$ satisfies $A.1$, $A.2$, $A.3$, $A.4(\alpha)$, $A.5(\alpha)$ and $A.7$.*
2. *There exists a C-independent, homogenous of degree 1, α-monotonic and α-concave functional $I : B_0(\Sigma) \to \mathbb{R}$ and affine $u : X \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$, $f \succcurlyeq g \iff I(u(f)) \geq I(u(g))$.*

Moreover, I is unique and u is unique up to positive affine transformations.

Proof. 1. $\implies$ 2.: First recall that under $A.1, A.3$ and $A.7$, $A.4(\alpha)$ implies $A.4$ as shown in Lemma 2. Thus Lemma 1 from Ghirardato, Maccheroni, and Marinacci (2004) can be used. It shows that $A.1 - A.4$ imply the existence and affinity of $u : X \to \mathbb{R}$ as well as the existence of the functional $I : B_0(\Sigma) \to \mathbb{R}$ which is monotonic, C-independent and homogenous of degree 1. To show α-monotonicity assume $a \geq b$. Since $\frac{1}{2}a + \frac{1}{2}(-a) = 0^* = \frac{1}{2}b + \frac{1}{2}(-b)$, $A.5(\alpha)$ and C-independence imply the required $\alpha I(a) + (1-\alpha)I(-a) \geq (\leq) \alpha I(b) + (1-\alpha)I(-b)$ for $\alpha > \frac{1}{2}$ ($\alpha < \frac{1}{2}$). To show α-concavity consider $a, b \in B_0(\Sigma)$ and assume

$$\alpha I(a) + (1-\alpha)I(-a) \geq \alpha I(b) + (1-\alpha)I(-b). \tag{22}$$

$A.5(\alpha)$ implies for all $\beta \in [0,1]$

$$\alpha I(\beta a + (1-\beta)b) + (1-\alpha)I(\beta(-a) + (1-\beta)(-b)) \geq \alpha I(b) + (1-\alpha)I(-b). \tag{23}$$

First assume that (22) holds with equality. Then

$$\alpha I(b) + (1-\alpha)I(-b) = \alpha(\beta I(a) + (1-\beta)I(b)) + (1-\alpha)((\beta I(-a) + (1-\beta)I(-b))),$$

which together with (23) shows the required. Now assume that (22) is strict. Consider $\gamma = \frac{\alpha I(a)+(1-\alpha)I(-a)-(\alpha I(b)+(1-\alpha)I(-b))}{2\alpha-1}$, which implies $\alpha\gamma + (1-\alpha)(-\gamma) = \alpha I(a) + (1-\alpha)I(-a) - (\alpha I(b) + (1-\alpha)I(-b))$. Define $c = b + \gamma^* \in B_0(\Sigma)$. By construction and C-independence we have

$$\alpha I(a) + (1-\alpha)I(-a) = \alpha I(b) + (1-\alpha)I(-b) + \alpha\gamma + (1-\alpha)(-\gamma) = \alpha I(c) + (1-\alpha)I(-c).$$

Utilizing the previous case and again C-independence we have

$$\begin{aligned}
&\alpha I(\beta a + (1-\beta)b) + (1-\alpha)I(\beta(-a) + (1-\beta)(-b)) + (1-\beta)(\alpha\gamma + (1-\alpha)(-\gamma)) \\
=&\alpha I(\beta a + (1-\beta)(b+\gamma^*)) + (1-\alpha)I(\beta(-a) + (1-\beta)(-(b+\gamma^*))) \\
=&\alpha I(\beta a + (1-\beta)c) + (1-\alpha)I(\beta(-a) + (1-\beta)(-c)) \\
\geq&\alpha(\beta I(a) + (1-\beta)I(c)) + (1-\alpha)(\beta I(-a) + (1-\beta)I(-c)) \\
=&\alpha(\beta I(a) + (1-\beta)I(b)) + (1-\alpha)((\beta I(-a) + (1-\beta)I(-b))) + (1-\beta)(\alpha\gamma + (1-\alpha)(-\gamma)).
\end{aligned}$$

Substracting $(1-\beta)(\alpha\gamma + (1-\alpha)(-\gamma))$ from both sides results in the required.

2. $\implies$ 1.: $A.1 - A.3$ follow from Lemma 1 in Ghirardato, Maccheroni, and Marinacci (2004), hereby using that α-monotonicity implies monotonicity as shown in the case $\alpha' = 1$ in the proof of Lemma 3.

Consider a duo of complementary pairs (f, f', g, g'). Due to C-independence of I we can assume without loss of generality that $u(\frac{1}{2}f + \frac{1}{2}f') = 0^* = u(\frac{1}{2}g + \frac{1}{2}g')$. Define $a = u(f)$ and $b = u(g)$, which implies $-a = u(f')$ and $-b = -u(g')$. Assume $\alpha > \frac{1}{2}$.

For $A.4(\alpha)$ assume that $f(s) \succcurlyeq g(s)$ for all $s \in S$ which implies $a \geq b$. α-monotonicity implies $\alpha I(a) + (1-\alpha)I(-a) \geq \alpha I(b) + (1-\alpha)I(-b)$. Thus $\alpha x_f + (1-\alpha)x_{f'} \succcurlyeq \alpha x_g + (1-\alpha)x_{g'}$.

For $A.5(\alpha)$ assume $\alpha x_f + (1-\alpha)x_{f'} \succcurlyeq \alpha x_g + (1-\alpha)x_{g'}$, i.e. $\alpha I(a) + (1-\alpha)I(-a) \geq$

$\alpha I(b) + (1-\alpha)I(-b)$. Since I is α-concave we have for all $\beta \in [0,1]$

$$\begin{aligned}
&\alpha I(\beta a + (1-\beta)b) + (1-\alpha)I(\beta(-a) + (1-\beta)(-b)) \\
\geq &\alpha\beta I(a) + \alpha(1-\beta)I(b) + (1-\alpha)\beta I(-a) + (1-\alpha)(1-\beta)I(-b) \\
= &\beta(\alpha I(a) + (1-\alpha)I(-a) + (1-\beta)(\alpha I(b) + (1-\alpha)I(-b) \\
\geq &\alpha I(b) + (1-\alpha)I(-b).
\end{aligned}$$

This implies the required $\alpha x_{\beta f+(1-\beta)g} + (1-\alpha)x_{\beta f'+(1-\beta)g'} \succcurlyeq \alpha x_g + (1-\alpha)x_{g'}$.

The case $\alpha < \frac{1}{2}$ is analogous.

□

For a functional $I : B_0(\Sigma) \to \mathbb{R}$ and $\alpha \in [0,1]\backslash\{\frac{1}{2}\}$ define $I^\alpha : B_0(\Sigma) \to \mathbb{R}$ by

$$I^\alpha(a) = \frac{\alpha I(a) + (1-\alpha)I(-a)}{2\alpha - 1} \tag{24}$$

for all $a \in B_0(\Sigma)$.

Through rearrangements this results in

$$I(a) = \alpha I^\alpha(a) - (1-\alpha)I^\alpha(-a). \tag{25}$$

The following lemma shows that a functional I satisfies the properties of Lemma 4 if and only if I^α satisfies the properties of Lemma 3.3 in Gilboa and Schmeidler (1989).

Lemma 5. *Let $\alpha \in [0,1]\backslash\{\frac{1}{2}\}$ and $I, I^\alpha : B_0(\Sigma) \to \mathbb{R}$ be functionals connected via (24). Then the following are equivalent:*

1. *I is C-independent, homogenous of degree 1, α-monotonic and α-concave.*

2. *I^α is C-independent, homogenous of degree 1, monotonic and concave.*

Proof. To show the claim for C-independence and homogeneity of degree 1 consider $a \in B_0(\Sigma)$, $c \in \mathbb{R}$ and $\gamma \geq 0$. First of all, for a functional $J : B_0(\Sigma) \to \mathbb{R}$ which satisfies C-independence we have $J(0^*) = J(0^* + 0^*) = J(0^*) + J(0^*)$ which implies $J(0^*) = 0$. Thus $0 = J(0^*) = J(c^* - c^*) = J(c^*) + J(-c^*)$, which implies $J(-c^*) = -J(c^*)$.

Assume that I satisfies C-independence and homogeneity of degree 1, i.e. $I(\gamma a+c^*) = \gamma I(a) + I(c^*)$. Using $I(-c^*) = -I(c^*)$ and (24) we get

$$
\begin{aligned}
I^\alpha(c^*) &= \frac{\alpha I(c^*) + (1-\alpha)I(-c^*)}{2\alpha - 1} \\
&= \frac{\alpha I(c^*) - (1-\alpha)I(c^*)}{2\alpha - 1} \\
&= I(c^*).
\end{aligned}
$$

Now

$$
\begin{aligned}
I^\alpha(\gamma a + c^*) &= \frac{\alpha I(\gamma a + c^*) + (1-\alpha)I(-(\gamma a + c^*)}{2\alpha - 1} \\
&= \frac{\alpha(\gamma I(a) + I(c^*)) + (1-\alpha)(\gamma I(-a) - I(c^*))}{2\alpha - 1} \\
&= \gamma \frac{\alpha I(a) + (1-\alpha)I(-a)}{2\alpha - 1} + I(c^*) \\
&= \gamma I^\alpha(a) + I^\alpha(c^*).
\end{aligned}
$$

Assume that I^α satisfies C-independence and homogeneity of degree 1, i.e. $I^\alpha(\gamma a + c^*) = \gamma I^\alpha(a) + I\alpha(c^*)$. Using $I^\alpha(-c^*) = -I^\alpha(c^*)$ and (25) we get

$$
\begin{aligned}
I(c^*) &= \alpha I^\alpha(c^*) - (1-\alpha)I^\alpha(-c^*) \\
&= \alpha I^\alpha(c^*) + (1-\alpha)I^\alpha(c^*) \\
&= I^\alpha(c^*).
\end{aligned}
$$

Now

$$
\begin{aligned}
I(\gamma a + c^*) &= \alpha I^\alpha(\gamma a + c^*) - (1-\alpha) I^\alpha(-(\gamma a + c^*) \\
&= \alpha(\gamma I^\alpha(a) + I^\alpha(c^*)) - (1-\alpha)(\gamma I^\alpha(-a) + I^\alpha(-c^*) \\
&= \gamma \left[\alpha \frac{\alpha I(a) + (1-\alpha) I(-a)}{2\alpha - 1} + (1-\alpha) \frac{\alpha I(-a) + (1-\alpha) I(a)}{2\alpha - 1} \right] \\
&\quad + (\alpha + (1-\alpha)) I^\alpha(c^*) \\
&= \gamma \frac{2\alpha - 1}{2\alpha - 1} I(a) + I(c^*) \\
&= \gamma I(a) + I(c^*).
\end{aligned}
$$

For the concavity properties consider $a, b \in B_0(\Sigma)$ and $\beta \in [0,1]$. Assume $\alpha > \frac{1}{2}$.

$$
\begin{aligned}
& I^\alpha(\beta a + (1-\beta) b) \geq \beta I^\alpha(a) + (1-\beta) I^\alpha(b) \\
\iff & \alpha I(\beta a + (1-\beta) b) + (1-\alpha) I(\beta(-a) + (1-\beta)(-b)) \\
& \geq \alpha\beta I(a) + \alpha(1-\beta) I(b) + (1-\alpha)\beta I(-a) + (1-\alpha)(1-\beta) I(-b),
\end{aligned}
$$

which shows the required. The case $\alpha < \frac{1}{2}$ is analogous.

For the monotonicity properties consider $a, b \in B_0(\Sigma)$ with $a \geq b$. Assume $\alpha > \frac{1}{2}$.

$$
\begin{aligned}
& I^\alpha(a) \geq I^\alpha(b) \\
\iff & \alpha I(a) + (1-\alpha) I(-a) \geq \alpha I(b) + (1-\alpha) I(-b).
\end{aligned}
$$

The case $\alpha < \frac{1}{2}$ is analogous.

□

The following lemma is needed for the subsequent lemma.

Lemma 6. *For all prior sets $\mathcal{C} \subseteq pc(\Sigma)$ and $a \in B_0(\Sigma)$*

$$
\min_{P \in \mathcal{C}} \int a dP = - \max_{P \in \mathcal{C}} \int -a dP. \tag{26}
$$

Proof. For all $a \in B_0(\Sigma)$ and $P \in pc(\Sigma)$ we have $\int a dP = -\int -a dP$. This implies

$\{\arg\min_{P\in\mathcal{C}} \int adP\} = \{\arg\max_{P\in\mathcal{C}} \int -adP\}$. Let $Q \in pc(\Sigma)$ be an element of this set. Then

$$\min_{P\in\mathcal{C}} \int adP + \max_{P\in\mathcal{C}} \int -adP = \int adQ + \int -adQ = \int a + (-a)dQ = \int 0^* dQ = 0.$$

$\square$

The following lemma shows that I has an α-MEU representation with prior set $\mathcal{C}$ if and only if I^α has an MEU representation with prior set $\mathcal{C}$.

Lemma 7. *Let $\alpha \in [0,1]\backslash\{\frac{1}{2}\}$ and $I, I^\alpha : B_0(\Sigma) \to \mathbb{R}$ be functionals connected via (24). Let $\mathcal{C} \subseteq pc(\Sigma)$ be a prior set. Then the following are equivalent:*

1. *I has an α-MEU representation with prior set $\mathcal{C}$.*
2. *I^α has an MEU representation with prior set $\mathcal{C}$.*

Proof. Assume that $I^\alpha(a) = \min_{P\in\mathcal{C}} \int adP$ for all $a \in B_0(\Sigma)$. Then

$$\begin{aligned}
\alpha \min_{P\in\mathcal{C}} \int adP + (1-\alpha)\max_{P\in\mathcal{C}} \int adP &\overset{(26)}{=} \alpha \min_{P\in\mathcal{C}} \int adP - (1-\alpha)\min_{P\in\mathcal{C}} \int -adP \\
&= \alpha \frac{\alpha I(a) + (1-\alpha)I(-a)}{2\alpha - 1} \\
&\quad - (1-\alpha)\frac{\alpha I(-a) + (1-\alpha)I(a)}{2\alpha - 1} \\
&= I(a)\frac{\alpha^2 - 1 + 2\alpha - \alpha^2}{2\alpha - 1} = I(a).
\end{aligned}$$

For the other direction assume that $I(a) = \alpha \min_{P\in\mathcal{C}} \int adP + (1-\alpha)\max_{P\in\mathcal{C}} \int adP$ for all $a \in B_0(\Sigma)$. Then

$$\begin{aligned}
I^\alpha(a) &= \frac{\alpha(\alpha \min_{P\in\mathcal{C}} \int adP + (1-\alpha)\max_{P\in\mathcal{C}} \int adP) + (1-\alpha)(\alpha \min_{P\in\mathcal{C}} \int -adP + (1-\alpha)\max_{P\in\mathcal{C}} \int -adP)}{2\alpha - 1} \\
&\overset{(26)}{=} \frac{\alpha(\alpha \min_{P\in\mathcal{C}} \int adP + (1-\alpha)\max_{P\in\mathcal{C}} \int adP) - (1-\alpha)(\alpha \max_{P\in\mathcal{C}} \int adP + (1-\alpha)\min_{P\in\mathcal{C}} \int adP)}{2\alpha - 1} \\
&= \min_{P\in\mathcal{C}} \int adP \frac{\alpha^2 - 1 + 2\alpha - \alpha^2}{2\alpha - 1} \\
&= \min_{P\in\mathcal{C}} \int adP.
\end{aligned}$$

$\square$

We can now prove the representation theorem.

Proof of Theorem 1. Let $\succcurlyeq$ be a binary relation on $\mathcal{F}$ satisfying $A.1 - A.4$ and assume $\alpha \in [0,1] \backslash \{\frac{1}{2}\}$. Let $I : B_0(\Sigma) \to \mathbb{R}$ be the functional induced by $\succcurlyeq$ through Lemma 1 in Ghirardato, Maccheroni, and Marinacci (2004). Let $I^\alpha : B_0(\Sigma) \to \mathbb{R}$ be defined as in (24) and $\succcurlyeq^\alpha$ the binary relation on $\mathcal{F}$ induced by I^α. We have:

$$
\begin{gathered}
\succcurlyeq \text{ satisfies } A.1, A.2, A.3, A.4(\alpha), A.5(\alpha), A.7 \\
\iff \text{(Lemma 4)} \\
I \text{ is C-independent, homogenous of degree 1, } \alpha\text{-monotonic, } \alpha\text{-concave} \\
\iff \text{(Lemma 5)} \\
I^\alpha \text{ is C-independent, homogenous of degree 1, monotonic, concave} \\
\iff \text{(Lemma 3.3, Gilboa and Schmeidler (1989))} \\
\succcurlyeq^\alpha \text{ satisfies } A.1, A.2, A.3, A.4, A.5. \\
\iff \text{(Theorem 1, Gilboa and Schmeidler (1989))} \\
\succcurlyeq^\alpha \text{ has a MEU representation.}
\end{gathered}
$$

Lemma 7 now suffices for the equivalence between 1. and 2. in the theorem. Furthermore the prior sets of the representation of $\succcurlyeq$ and $\succcurlyeq^\alpha$ coincide.

The uniqueness properties follow from the uniqueness properties in Theorem 1 of Gilboa and Schmeidler (1989) as I^α is well-defined.

□

6.3 Proof of Corollary 1 and Proposition 1

Proof of Corollary 1. Let $\succcurlyeq$ be a binary relation on $\mathcal{F}$. First assume that $\succcurlyeq$ has an α-MEU representation with a singleton prior set $\mathcal{C} = \{P\}$ with $P \in \Delta(S, \Sigma)$. Then it is immediate that every $\alpha \in [0,1]$ allows a representation with prior set $\{P\}$ as for all acts $f \in \mathcal{F}$, $\max_{P \in \mathcal{C}} \int u(f)dP = \min_{P \in \mathcal{C}} \int u(f)dP = \int u(f)dP$.

So now assume that $\succcurlyeq$ has some α-MEU representation but none with a singleton prior set. Proposition 1 in Frick, Iijima, and Yaouanq (2020) implies the mutual exclusiveness of three cases: the α-values that allow a representation are all in either $[0, \frac{1}{2})$, $(\frac{1}{2}, 1]$ or $\{\frac{1}{2}\}$. The latter case is trivial, thus assume that there is a representation with $\alpha \neq \frac{1}{2}$. We first show that the set of α-values that allow a representation is convex. Assume that α_1 and α_2 allow a representation which implies either $\alpha_1, \alpha_2 > \frac{1}{2}$ or $\alpha_1, \alpha_2 < \frac{1}{2}$. Theorem 1 implies that $\succcurlyeq$ satisfies $A.4(\alpha_1)$ and $A.5(\alpha_1)$ as well as $A.4(\alpha_2)$ and $A.5(\alpha_2)$. Lemma 3 implies that $\succcurlyeq$ also satisfies $A.4(\alpha_3)$ and $A.5(\alpha_3)$ for all α_3 between α_1 and α_2. Theorem 1 implies the required convexity.

It remains to be shown that the interval is closed. Assume that the interval is a subset of $(\frac{1}{2}, 1]$, the case where it is inside $[0, \frac{1}{2})$ being analogous.

Assume that $\alpha_i \searrow \alpha$ and that for all $i \in \mathbb{N}$, α_i allows a representation, i.e. $A.4(\alpha_i)$ and $A.5(\alpha_i)$ hold for all $i \in \mathbb{N}$. We must show that $A.4(\alpha)$ and $A.5(\alpha)$ hold. Lemma 3 implies that $A.5(\alpha)$ holds. For $A.4(\alpha)$ it suffices to show that the functional satisfies α-monotonicity:

$$\begin{aligned}\alpha(V(f) - V(g)) + (1-\alpha)(V(f') - V(g')) &= \lim_{i\to\infty} \underbrace{\alpha_i(V(f) - V(g)) + (1-\alpha_i)(V(f') - V(g'))}_{\geq 0 \ \forall i \in \mathbb{N}} \\ &\geq 0.\end{aligned}$$

Assume that $\alpha_i \nearrow \alpha$ and that for all $i \in \mathbb{N}$, α_i allows a representation, i.e. $A.4(\alpha_i)$ and $A.5(\alpha_i)$ hold for all $i \in \mathbb{N}$. We must show that $A.4(\alpha)$ and $A.5(\alpha)$ hold. Lemma 3 implies that $A.4(\alpha)$ holds. For $A.5(\alpha)$ it suffices to show that the functional satisfies α-convavity:

$$\begin{aligned}&\alpha V(\beta f + (1-\beta)g) + (1-\alpha)V(\beta f' + (1-\beta)g') \\ =&\lim_{i\to\infty} \alpha_i V(\beta f + (1-\beta)g) + (1-\alpha_i)V(\beta f' + (1-\beta)g') \\ \geq&\lim_{i\to\infty} \alpha_i(\beta V(f) + (1-\beta)V(g) + (1-\alpha_i)(\beta V(f') + (1-\beta)V(g')) \\ =&\alpha(\beta V(f) + (1-\beta)V(g) + (1-\alpha)(\beta V(f') + (1-\beta)V(g')).\end{aligned}$$

$\square$

Proof of Proposition 1. Let $[\underline{\alpha}, \overline{\alpha}]$ be the interval derived in Corollary 1. Assume $\underline{\alpha} > \frac{1}{2}$ (the case $\overline{\alpha} < \frac{1}{2}$ being analogous) and let $\overline{\mathcal{C}}$ be the corresponding prior set.

Assume for contradiction that $\overline{\mathcal{C}}$ does not intersect the boundary of $\Delta(S, \Sigma)$. Then there exists a sufficiently small $\gamma > 1$ such that the γ-expansion of $\overline{\mathcal{C}}$, $\mathcal{C}^* = \{\gamma P + (1-\gamma)Q | P, Q \in \overline{\mathcal{C}}\}$, also does not intersect the boundary of $\Delta(S, \Sigma)$. In particular $\mathcal{C}^* \subseteq \Delta(S, \Sigma)$. Proposition 1 in Frick, Iijima, and Yaouanq (2020) implies that there exists an α^* such that $\frac{1}{2} < \alpha^* < \underline{\alpha}$ and preferences have an α^*-MEU representation with prior set $\mathcal{C}^*$. Theorem 1 implies that $A.4(\alpha^*)$ holds, a contradiction to $\underline{\alpha}$ being the smallest such value.

$\square$

6.4 Proof of Proposition 2

Proof of Proposition 2. Consider $f, g, h \in \mathcal{F}$, $\alpha \in [0, 1]$ and let x_f, x_g, x_h be constant equivalents of f, g and h, respectively.

First assume $A.9$. From $f \sim x_f$ it implies $\alpha f + (1-\alpha)x_g \sim \alpha x_f + (1-\alpha)x_g$. From $g \sim x_g$ it implies $\alpha f + (1-\alpha)g \sim \alpha f + (1-\alpha)x_g$. Transitivity implies the required $\alpha f + (1-\alpha)g \sim \alpha x_f + (1-\alpha)x_g$.

Now assume that equation (10) holds for all acts and that risk independence holds, i.e. for all $x, y, z \in X$ and $\alpha \in (0, 1)$ we have $x \succcurlyeq y \iff \alpha x + (1-\alpha)z \succcurlyeq \alpha y + (1-\alpha)z$. Assume $f \succcurlyeq g$. Due to $A.1$ this is equivalent to $x_f \succcurlyeq x_g$. Risk independence implies $\alpha x_f + (1-\alpha)x_h \succcurlyeq \alpha x_g + (1-\alpha)x_h$. Using (10) twice as well as $A.1$ we conclude the required $\alpha f + (1-\alpha)h \succcurlyeq \alpha g + (1-\alpha)h$.

$\square$

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