# Liquidity Hoarding and Lotteries*

Benjamin Müller†
Master Thesis - Supervised by Prof. Aleksander Berentsen
University of Basel

January 8, 2012

**Abstract**

This paper extends the interbank market model by Gale and Yorulmazer (2011) and introduces randomized trading schemes in the form of lotteries. Gale and Yorulmazer (2011) study a model of liquidity management in general equilibrium and show that banks are hoarding liquidity due to precautionary as well as speculative motives. Their model exhibits two sources of inefficiencies. First, banks hoard too much liquidity. Second, banks' portfolio choice features insufficient liquidity holdings. The motivation of this paper is to show whether the introduction of lotteries mitigates or even eliminates the identified inefficiencies. I show that lotteries can eliminate the inefficient hoarding of liquidity and mitigate the inefficient portfolio choice. This paper is related to impaired interbank markets and liquidity hoarding by banks in the current debt crisis as well as in the recent financial crisis.

*I would like to thank Prof. Aleksander Berentsen for the supervision of my master thesis. I benefited a lot from his comments and the discussions in the course of the preparation of this paper. Moreover, many thanks go to Anja Frommherz, Marco Poltera and Simon Schmid for proofreading this paper.

†Student number: 06-060-230; Email: benji.mueller@stud.unibas.ch

# Contents



# List of Figures

# 1 Introduction

The recent financial crisis has shown the importance of a well-functioning interbank market. According to Brunnermeier (2009), the interbank market's inability to distribute liquidity efficiently was an important feature of the financial crisis. In normal times, the interbank market is highly liquid and channels liquidity from banks and institutions holding excess liquidity to banks and institutions which are in need of liquidity. According to Heider, Hoerova and Holthausen (2009), a well-functioning interbank market is important in three dimensions. First, the liquidity management of banks and institutions heavily depends on the possibility to balance their liquidity holdings. Second, the interbank market has an important role for the transmission of monetary policy.[1] Third, many economically important prices and interest rates are derived from benchmark rates in the interbank market.[2] In the course of the latest financial crisis but also in the current debt crisis the functioning of the interbank market became heavily impaired. On the one hand, banks reported their inability to borrow in the interbank market due to record high levels of borrowing rates and on the other hand, recurring freezing of the interbank market was observed. As a result, many central banks offered unlimited borrowing facilities and, therefore, became the main counterparty for many banks and financial institutions.[3]

According to Gale and Yorulmazer (2011), there are two main explanations for these observations which both are motivated by information asymmetry. The first explanation is the counter-party risk. Each bank has private information about the risks carried on its books but cannot observe those of the other banks. Huge exposures to sub-prime asset-backed securities during the financial crisis as well as to government debt of highly indebted states in the current debt crisis led to a stop of lending activity because of the fear that the counterparty might be at risk. The second explanation is related to the first one. Banks who are financially sound can be perceived as highly risky just because of rumors and doubts. As a consequence, every bank may fear to lose access to liquidity and, therefore, hoards liquidity.

This paper is based on the model of Gale and Yorulmazer (2011) which studies liquidity management in general equilibrium in a four period model where banks exchange illiquid assets for liquid assets in an interbank market. The set up of the model allows identifying two motives why banks hoard liquidity. The first motive is a precautionary one. Bankers may receive a liquidity shock in the future and although they can obtain cash in exchange for assets, the price may be very high. In that case, bankers prefer to hoard liquidity

[1] For example, the Swiss National Bank implements its monetary policy by steering the three month Swiss franc LIBOR (London interbank offered rate).

[2] For instance, the LIBOR represents an important benchmark to price mortgages in many countries.

[3] Indications for this are on the one hand, the massive deposits of banks held at the central bank and on the other hand, the massive demand for central bank liquidity by the market.

instead of acquiring it in the next period. The second motive is a speculative one. If the demand for cash in the next period is very high, the price for cash will be very high as well. Therefore, bankers can make a profit if they hoard liquidity in this period and sell it at a high price in the next period. Gale and Yorulmazer (2011) identify two inefficiencies in their model. First, they find that too many bankers are hoarding cash instead of supplying it in the market. Second, they detect that too many bankers are choosing a portfolio with insufficient cash holdings. In order to eliminate the first inefficiency, the authors suggest direct lending facilities at the central bank.[4] In order to eliminate the second inefficiency, they propose regulatory interventions by the supervisory authorities. One approach would be to specify liquidity requirements as in Basel III. The novelty of Gale and Yorulmazer (2011) is to study the portfolio choice as well as liquidity hoarding in a unified framework.

The motivation of this paper is to show a different approach to confront the inefficiencies identified by Gale and Yorulmazer (2011). I use a well known idea from the theoretical literature to encounter nonconvexitites and indivisibilities and introduce randomized trading schemes in the form of lotteries. I will show that the introduction of lotteries in the second period can eliminate inefficient hoarding and induce markets to clear in any case. In order to study the second inefficiency, I consider a simplified model without hoarding and a distributional assumption regarding the shocks. This allows me to compare the portfolio choice to the laisser-faire equilibrium by Gale and Yorulmazer (2011) as well as to the social planners' solution. The conclusion is that the introduction of lotteries can at least mitigate the inefficient portfolio choice. Consequently, this paper suggests little robustness of the results by Gale and Yorulmazer (2011).

The structure of this paper is as follows. Section 2 summarizes the literature on liquidity hoarding, introduces the idea of lotteries based on the paper by Berentsen, Molico and Wright (2001) and presents the basic model by Gale and Yorulmazer (2011). In Section 3 I set up the model with lotteries and derive the market clearing conditions. Section 4 presents the implications of lotteries regarding the portfolio choice. Finally, Section 5 concludes.

[4] According to Goodfriend and King (1988), an efficient interbank market will distribute liquidity among market participants. However, as Gale and Yorulmazer (2011) report, this does not necessarily work. The authors refer to the statements of Mervyn King (Governor of the Bank of England) and Alistair Darling (Chancellor of the Exchequer) at the hearings concerning the Northern Rock case in Fall 2007. Northern Rock had to borrow £14 billion directly from the Bank of England. If Northern Rock had tried to get the same amount in the interbank market, the Bank of England would have been required to inject many more billions via open market operations. As a consequence, direct lending may represent a more efficient solution.

# 2 Literature

This section discusses the related literature. The first part deals with the empirical and theoretical literature on liquidity hoarding and lotteries. The second part presents the basic model by Gale and Yorulmazer (2011).

## 2.1 Literature on liquidity

Since the collapse of Lehman Brothers and the outbreak of the financial crisis in September 2007, the literature on liquidity has increased significantly.[5] A major part of the literature on liquidity is based on theoretical models and can be separated in three dimensions: the endogenous choice of liquidity (portfolio choice), interbank markets and banks' free-riding on others' liquidity. Nevertheless, many papers provide empirical evidence for liquidity hoarding in the interbank market. At a general level, this paper is related to the idea of financially constrained buyers of assets and prices determined by the available liquidity in a market. This literature is reviewed in Allen and Gale (2005).

**Empirical evidence:** Heider et al. (2008) analyze the unsecured euro interbank market and provide evidence for liquidity hoarding. They consider the spread between the three-month unsecured interest rate and the overnight index swap in three months' time[6] and analyze the co-movement with the amount of excess reserves banks hold with the European Central Bank (ECB). In normal times, banks prefer to lend in the interbank market instead of holding excess reserves at the ECB since the rates at the latter are punitive relative to the market rates. The authors observe that an increased spread, indicating increased tension in the interbank market, is positively correlated with excess reserve holdings at the ECB. This is interpreted as evidence for liquidity hoarding. Acharya and Merrouche (2009) analyze the money demand by large UK banks before and during the financial crisis and study its implications on the UK Sterling money market. The authors observe an average increase of liquidity holdings by 30% after the freezing of money markets following August 9th, 2007.[7] Moreover, the authors find a positive relation between liquidity holdings and calendar days with large amounts of payment activity. Ashcraft, McAndrews and Skeie (2008) used intraday data of the Federal Reserve Bank and Fedwire interbank transactions to estimate all overnight Fed Funds trades during the period of September 2007 until August 2008 and find evidence for precautionary hoard-

[5] A quick check at ideas.repec.org indicates that the key word "liquidity" exhibits more hits in the period from 2007 until the end of 2011 than in the period from 1950 until 2007.

[6] An interest rate which reflects expectations about the three-month overnight unsecured rate.

[7] On the August 9th, 2007 the Dow Jones Industrial fell by almost 400 points and the major central banks all around the world injected money into the markets due to increased economic concerns.

ing.[8] Afonso, Kovner and Schoar (2010) examine US Fed Funds markets after the Lehman Brothers collapse and find that increased concerns about counterparty risks goes hand in hand with reduced liquidity and increased financing costs for weaker banks. However, the authors find no evidence for liquidity hoarding in the overnight Fed Funds market. Ivashina and Scharfstein (2008) reveal significant rationing of bank lending during the financial crisis. Finally, Guggenheim, Kraenzlin and Schumacher (2011) developed an algorithm to identify transactions and market rates for the unsecured Swiss franc money market. They can detect a freezing of the market for several months after the collapse of Lehman Brothers. Moreover, they observe a flight from the unsecured to the secured interbank money market during the financial crisis.

The empirical literature described so far only deals with the recent financial crisis but not with the current debt crisis. Nevertheless, triggered by the current debt crisis, these days the situation at the interbank market is as impaired as during the financial crisis.[9]

**Portfolio choice:** Acharya, Shin and Yorulmazer (2009) show that the choice of bank liquidity is counter-cyclical to the economic condition. More precisely, they show that liquidity holdings are inefficiently low during economic booms and excessively high during crisis. They argue that potential profits from acquiring assets at fire-sale prices in the future make it attractive to hoard liquidity. As a result, Acharya et al. (2009) suggest to support banks only conditionally on certain liquidity requirements. Banks which do not fulfill these requirements will not get access to central bank liquidity. Allen and Gale (2004) develop a general equilibrium model where banks endogenously choose the level of liquid assets. In contrast to the seminal paper by Diamond and Dybvig (1983), they study the intermediation of complex financial systems instead of a single bank. Acharya and Skeie (2010) present a model where leveraged banks demand liquidity for precautionary reasons and, by this, produce extreme rates for term interbank loans and a reduced volume of trade. Hence, the authors provide explanations for the increase in spreads and the collapse in maturities in the term interbank market observed during the financial crisis.

**Interbank market:** In the literature on interbank markets two different views can be identified. The first view is expressed by Goodfriend and King (1988). With efficient interbank markets, they propagate that central banks should provide liquidity only

[8] Fedwire is the real-time electronic gross settlement system provided by the Federal Reserve Banks for large institutions and banks.

[9] On 26th November, 2011 The Economist 401(8761) p. 15-16 reports: 'The panic engulfing Europe's banks is no less alarming. The access to wholesale funding markets has dried up, and the interbank market is increasingly stressed, as banks refuse to lend to each other. Firms are pulling deposits from peripheral countries' banks. This backdoor run is forcing banks to sell assets and squeeze lending; the credit crunch could be deeper than the one Europe suffered after Lehman Brothers collapsed'.

through open market operations and not by lending to individual banks. The interbank market will then allocate the liquidity in an efficient way. The second view propagates the idea that the interbank market cannot allocate liquidity efficiently due to frictions (mainly due to information asymmetries). According to Flannery (1996), the potential loss of confidence in each others' counterparties' ability to pay can have dramatic effects on the interbank market. Therefore, the author proposes the idea of the central bank as a lender of last resort which lends directly to banks which are illiquid, but solvent. Freixas and Jorge (2007) study the role of interbank markets for the transmission of monetary policy. In particular, they analyze the impact of asymmetric information in the interbank market and identify credit rationing in equilibrium.

**Free-riding on others' liquidity:** The literature on free-riding on each others' liquidity has some interesting implications regarding the portfolio choice at the initial period. In the basic model of Gale and Yorulmazer (2011), banks choose an inefficiently low amount of liquidity holdings. An interpretation for this could be that bankers become illiquid and rely (free-ride) on the others' decision to remain liquid since they can get liquidity at a later stage of the model. Bhattacharya and Gale (1987) model interbank coordination by using the framework developed by Diamond and Dybvig (1983). They show that banks underinvest in liquid assets because they rely on ex post financing in the interbank market. In aggregate, the interbank market works inefficiently due to the individual underinvestment. Berentsen, Huber and Marchesiani (2011) extend the framework introduced by Berentsen and Waller (2011) and present an infinite horizon, dynamic general equilibrium model where financial market participants free-ride on liquidity. The model deals with idiosyncratic shocks and, therefore, differs from the standard literature which assumes an aggregate shock. Hence, it is not a model of crisis. The authors show that liquidity shortage also occurs in normal times. An important novelty of their contribution is the endogenous determination of the value of money due to the infinite horizon, whereas the literature so far only considered a fixed amount of periods, usually three. Their policy analysis shows that a restricted access to secondary financial markets can be welfare improving.

## 2.2 Literature on lotteries

The motivation of this paper is to show whether the equilibrium outcome in Gale and Yorulmazer (2011) can be improved by the introduction of randomized trading schemes in the form of lotteries. The implementation of lotteries in this paper is derived from Berentsen et al. (2001) who introduce lotteries into search-theoretic models of money. The authors describe an environment with two goods: money and goods and two types of agents: buyers (agents with money) and sellers (agents with goods) who meet randomly and bargain over lotteries (the joint probability distribution of goods and money). They assume money to be indivisible but consider the cases of indivisible as well as divisible goods. The authors introduce lotteries over cash to encounter the indivisible character of money and argue that

> It is well known from the study of various economic environments with indivisibilities and other nonconvexities that agents can often do better using randomized rather than deterministic trading mechanisms [...]. Berentsen et al. (2001:71)

First, the case with indivisible goods and lotteries over money is considered where the authors show that a monetary equilibrium exists if buyers have a bargaining power above a certain threshold. Thus, if a buyer meets a seller with a good he wants, goods trade with probability one and money with probability $\tau$ where $\tau = 1$. If the bargaining power is even larger, money trades with probability $\tau < 1$. Note that the introduction of lotteries provides a notion of prices since $\tau$ is the average amount of currency that trades for a unit of the good.

Second, the case with divisible goods ($q$) and lotteries over money is discussed. Berentsen et al. (2001) show that in this case, a unique monetary equilibrium for all parameters exists. Thus, if a buyer meets a seller with a good he wants, the buyer receives $q$ units of the good for sure but money is exchanged with a probability $\tau$ if the bargaining power is above a threshold level. Notice that $q$ is deterministic and independent of the delivery of money. Again, the authors find a role for lotteries even though goods are perfectly divisible. Their paper shows that in the divisible goods model welfare is higher, and strictly higher for some parameters, if lotteries are allowed. However, in the indivisible goods model this is not necessarily true for certain values of the bargaining power.

I will use a related approach to Berentsen et al. (2001) in order to introduce lotteries. Similar to their model, I establish on an indivisible good and a divisible good. I will introduce lotteries over the indivisible good and study its implications on the equilibrium allocation. However, since there is no bilateral trade in my model, bargaining will not be taken into account.

## 2.3 The basic model

Gale and Yorulmazer (2011) set up a four period model to examine liquidity management in general equilibrium. There are two types of assets: liquid assets (cash) and illiquid assets (asset) and two types of agents: creditors and bankers. Initially, bankers are endowed with one unit of cash and one unit of the asset. At date 0, bankers are characterized by their decision whether to spend their cash holdings (illiquid bankers) or to remain liquid (liquid bankers). There is an incentive to consume cash at date 0 since there is a utility cost of holding cash. At date 1 and 2, an interbank market opens where bankers can exchange assets for cash. Moreover, at date 1 and 2, bankers may receive random liquidity shocks which force them to discharge their debt of one unit of cash. Gale and Yorulmazer (2011:2) motivate the liquidity shocks by '[...] the demand of payment of a senior debt that can only be discharged by delivery of cash'. Note that debt can only be discharged using cash. The price in order to exchange assets for cash is defined by the amount of assets in units of cash.

Illiquid bankers who receive a shock need to obtain cash and, therefore, represent the demanders for cash at date 1. The supply of cash may come from liquid bankers without a shock. Gale and Yorulmazer (2011) present two reasons why a liquid banker may not want to supply cash. First, he may receive a liquidity shock at date 2 and, to discharge his debt, he would need to obtain cash for a potentially higher price. This characterizes the precautionary motive to hoard. Second, a liquid buyer may not want to supply cash because the demand and, therefore, the price at date 2 are high. As a result, it is attractive to hoard and use the cash to buy assets at date 2. This characterizes the speculative motive. At date 2, there are two groups of bankers who demand cash. Since one group acquired assets in the previous period, it is able to pay a higher price in order to get one unit of cash. Note that the different prices allow modeling the motives to hoard liquidity. Finally, at date 3, returns are realized and bankers who did not receive a shock so far need to discharge their debt.

Gale and Yorulmazer (2011) begin their analysis by characterizing the constrained-efficient allocation as the solution to a planner's problem. The planner is able to accumulate $m_0$ units of cash before the model is initialized at date 0. Note that the planner accumulates $m_0$ until the marginal cost is equal to the marginal value of cash.[10] At date 1 and 2, the planner can distribute cash to bankers which are in need of cash. However, he cannot reallocate assets among banks.[11] Cash is provided to all bankers in need of cash until the supply is exhausted. If there is some remaining supply after date 1, it will be forwarded to date 2. The constrained-efficient allocation is efficient since there is no inefficient hoarding regarding the prospective at date 2. As a result, liquidity hoarding takes place only at date 1.

In a next step, Gale and Yorulmazer (2011) characterize the laisser-faire market equilibrium where they identify two inefficiencies. First, the laisser-faire equilibrium exhibits

[10] This rules out the trivial solution of just accumulating an infinite amount of $m_0$.

[11] Note that this would imply a trivial solution to provide cash to those bankers receiving a shock and to provide the asset to those bankers with no shock. The model in this form would not lead to any frictions or inefficiencies.

inefficient hoarding of liquidity at date 1. This is due to the fact that hoarders do not internalize the welfare losses resulting from early defaults. Thus, in contrast to the social planner's problem, the marginal value of cash can be larger in the future than today which leads to an incentive to hoard. Second, comparing the portfolio choice of bankers in the laisser-faire economy with the social optimum shows that too many bankers choose to become illiquid at date 0. This is because bankers do not internalize the social value of paying off their debt.

Gale and Yorulmazer (2011) identify the incompleteness of markets as the fundamental reason for the inefficiency of the laisser-faire equilibrium. An efficient mechanism would allow entering into a contingent forward contract ex ante. In the case of a shock, the illiquid banker would obtain cash for a fixed price. Nevertheless, Gale and Yorulmazer (2011) prove that with the presence of asymmetric information, the laisser-faire market equilibrium cannot be improved. The authors present three approaches to solve the inefficiencies identified in the model. First, they show that a central bank can implement the constrained-efficient allocation from the planner's problem. However, the important (and extreme) assumption is that the central bank is the sole provider of liquidity. The second approach deals with the inefficiency of hoarding at date 1 and aims at facilitating lending at date 1 or direct lending via open market operations. The third approach considers the inefficient choice of liquidity holdings at date 0. Here, the authors bring up the idea of regulatory intervention by introducing liquidity requirements as in Basel III.

Finally, the paper provides comparative static analysis and shows how the equilibrium allocation and the allocation of a social planner will be affected by changes in expectations of future liquidity shocks as well as increased uncertainty and volatility of the shocks. The authors show that there is more hoarding if the expectation of future liquidity shocks increases. Moreover, the difference between the socially optimal level and the market solution increases.

# 3 The model with lotteries

This section introduces lotteries into the basic model by Gale and Yorulmazer (2011). First, I show the basics of the model and describe the activities in the four periods. Then, I derive the market clearing conditions. Finally, the equilibrium is presented.

## 3.1 Basics

**Assets:** There are two types of assets in this economy. There is a liquid asset called cash which is indivisible and has a return of one unit of consumption at date 3. In addition, there is an illiquid asset called asset which is divisible and has a return of $R > 1$ units of consumption at date 3.

**Agents:** There is a continuum of identical and risk neutral creditors indexed by $j \in [0, 1]$ and a continuum of identical and risk neutral bankers indexed by $i \in [0, 1]$. Each creditor $j$ is owed a debt of one unit of cash by bank $i = j$ which has to be discharged on demand. Creditors are assumed to consume their one unit of cash at one date. With probability $\theta_1$ they want to consume at date 1. With probability $(1 - \theta_1)\theta_2$ they want to consume at date 2. Finally, with probability $(1 - \theta_1)(1 - \theta_2)$ they consume at date 3. The creditor's expected utility function is

$$V(c_1, c_2, c_3) = \theta_1 c_1 + (1 - \theta_1)\theta_2 c_2 + (1 - \theta_1)(1 - \theta_2)c_3.$$

Bankers are endowed with one unit of the asset at date 0. Altogether, the portfolio of a banker at the beginning of date 0 comprises one unit of the asset and one unit of cash $(1, 1)$. Bankers can either consume their cash holdings at date 0 or at date 3 when the returns are realized. Given that there is a utility cost of holding cash $\rho > 1$ bankers prefer to consume at date 0. The banker's utility function is

$$U(c_0, c_3) = \rho c_0 + c_3.$$

It is important to note that consumption refers to eating up cash. Moreover, note that consumption and discharging debt implies that liquidity is taken out of the interbank market.

**Time:** Time is divided into four periods $(t = 0, 1, 2, 3)$. At date 0, bankers decide whether they consume their cash holdings or not. At date 1 and 2, the interbank market opens and bankers may receive random liquidity shocks which force them to discharge their debt. Since debt can only be discharged using cash, illiquid bankers need to exchange assets for cash first. If bankers cannot discharge their debt, they are forced to default in which case their payoff is 0.[12] At date 3, returns are realized and bankers who did not receive a shock so far have to discharge their debt. Then the economy ends.

**Distributions:** Liquidity shocks are random variables and are realized either at date 1 or at date 2. A shock at date $t$ is called $\theta_t$ and has a density function $f(\theta_t)$ and a cumulative distribution function $F(\theta_t)$, where $t = 1, 2$. $\theta_1$ and $\theta_2$ are *iid* with support $[0, 1]$. Since each creditor is owed a debt by one bank, bankers can only receive a shock once.

[12] The implicit assumption is that the liquidation costs are exactly equal to the remaining value of the portfolio.

**Lotteries:** A lottery is described by a probability $\tau \in (0, 1]$ at which cash is exchanged in trade. Note that the asset is always traded with probability one. I assume that lotteries are only possible at date 1 and later, I will show the endogenous choice of lotteries depending on the price at date 1. Note that I rule out the case $\tau = 0$ because I assume that no banker in need of cash would accept a lottery where he receives no cash for sure.

**Prices:** Prices in the interbank market at date 1 and 2 are defined by the amount of assets in units of cash. Since lotteries are ruled out at date 2, the price is defined as $p_2(\theta_1, \theta_2) = \frac{x_2(\theta_1,\theta_2)}{1}$, where $x_2(\theta_1, \theta_2)$ denotes the measure of assets. Note that the price at date 2 reflects the amount of assets which need to be transferred in order to get one unit of cash. The price at date 1 is defined by $p_1(\theta_1) = \frac{x_1(\theta_1)}{\tau}$, where $x_1(\theta_1)$, again, denotes the measure of assets. Now, cash is only transferred with probability $\tau$ whereas, with probability $(1 - \tau)$, no cash is transferred. Setting $\tau = 1$ refers to the basic model where cash is transferred with certainty.

At date 1, the price without lotteries a banker is able to pay is restricted to one since he only holds one unit of the asset. If the price is greater than one, the banker is unable to obtain cash and is forced to default. However, with lotteries this restriction of the price is not binding since $\tau \in (0, 1)$.

**Market clearing:** Markets clear if the amount of bankers who offer cash for assets is equal to the amount of bankers who demand cash for assets.

## 3.2 Activities

**Date 0:** Bankers initially are endowed with a portfolio $(1, 1)$. At date 0, they choose whether to consume their cash or not. Bankers who consume their cash are called illiquid bankers and those who keep it, liquid bankers. Let $0 \leq \alpha \leq 1$ denote the measure of illiquid bankers and $(1 - \alpha)$ the measure of liquid bankers. The $\alpha$ illiquid bankers end the period with a portfolio $(1, 0)$ and the $(1 - \alpha)$ liquid bankers end the period with a portfolio $(1, 1)$.

**Date 1:** At the beginning of date 1, a fraction $\theta_1$ of bankers receives a liquidity shock. The $(1 - \alpha)\theta_1$ liquid bankers who receive a shock can discharge their debt using their cash and end the period with a portfolio $(1, 0)$. Alternatively, they default and end the period with a portfolio $(0, 0)$. The $\alpha(1 - \theta_1)$ illiquid bankers who receive no shock do not trade and end the period with a portfolio $(1, 0)$. The $(1 - \alpha)(1 - \theta_1)$ liquid bankers who receive no shock can either become buyers or hoarders. $(1 - \alpha)(1 - \theta_1)(1 - \lambda)$ bankers become hoarders and end the period with a portfolio $(1, 1)$. The remaining two groups of bankers represent the demand and the supply of cash. The demand for cash comes from the $\alpha\theta_1$ illiquid bankers who receive a shock. In order to discharge their debt, they need to exchange a fraction of their assets for cash. The supply of cash comes from the $(1-\alpha)(1-\theta_1)\lambda$ liquid bankers without a shock. They hold unneeded cash which they offer in exchange for assets. Since trade with lotteries is possible, the two groups end the period with the following portfolios. Buyers receive $x_1(\theta_1)$ units of assets for sure, need to deliver cash with probability $\tau$ and can keep their unit of cash with probability $(1-\tau)$. With probability $\tau$, buyers end the period with a portfolio of $(1+x_1(\theta_1), 0)$ and with probability $(1-\tau)$ they end the period with a portfolio $(1+x_1(\theta_1), 1)$. Their expected portfolio at the end of date 1 is $(1+x_1(\theta_1), 1-\tau)$. Bankers, who demand cash transfer $x_1(\theta_1)$ units of assets for sure, receive cash with $\tau$ and receive no cash with probability $(1-\tau)$. Since they need to discharge their debt, they are forced to default if they get no cash. As a result, with probability $(1-\tau)$ they end the period with a portfolio $(0,0)$. With probability $\tau$ they receive cash but use it to discharge their debt and, therefore, end the period with a portfolio $(1-x_1(\theta_1), 0)$. Their expected portfolio at the end of date 1 is $([1-x_1(\theta_1)]\,\tau, 0)$. Figure 1 illustrates the allocations at date 0 and 1.

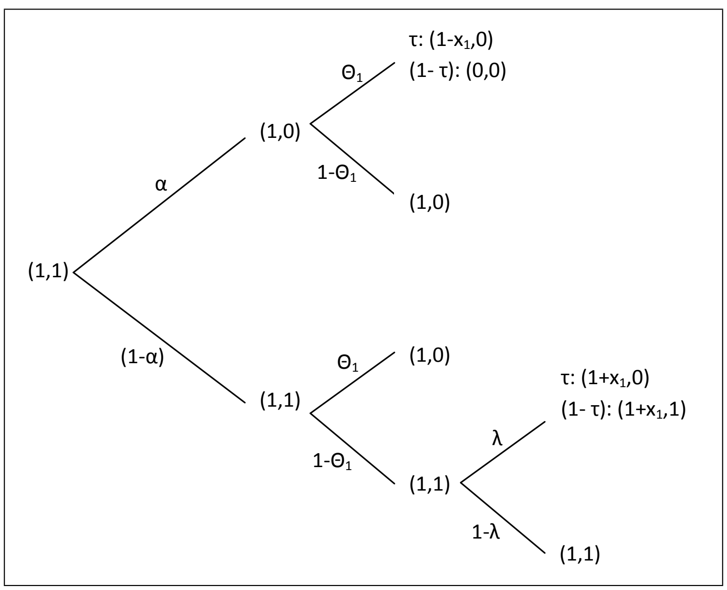


Figure 1: Allocations at date 0 and 1

**Date 2:** At the beginning of date 2, a fraction $\theta_2$ of bankers without a shock at date 1 receives a shock. Bankers who received a shock in the previous period are inactive at date 2 since they hold no cash and, therefore, will not participate in trade. Remember that lotteries are ruled out at date 2.

Two cases need to be considered regarding the $(1-\alpha)(1-\theta_1)\lambda\theta_2$ buyers who receive a liquidity shock at date 2. With probability $\tau$, there are $(1-\alpha)(1-\theta_1)\lambda\theta_2\tau$ bankers who need to exchange a fraction of their assets for cash. It will be optimal for them to do so as long as $x_2(\theta_1, \theta_2) \leq 1 + x_1(\theta_1)$. This group of bankers ends the period with a portfolio $(1 + x_1(\theta_1) - x_2(\theta_1, \theta_2), 0)$. With probability $(1-\tau)$ there are $(1-\alpha)(1-\theta_1)\lambda\theta_2(1-\tau)$ bankers who can discharge their debt using their cash holdings. Remember that this group of bankers did not have to transfer cash at date 1 and, therefore, can use it now. This group ends the period with a portfolio $(1 + x_1(\theta_1), 0)$. The expected portfolio of buyers with a shock at date 2 is $(1 + x_1(\theta_1) - x_2(\theta_1, \theta_2)\tau, 0)$.

Again, two cases need to be considered regarding the $(1-\alpha)(1-\theta_1)\lambda(1-\theta_2)$ buyers who do not receive a shock at date 2. With probability $\tau$ there are $(1-\alpha)(1-\theta_1)\lambda(1-\theta_2)\tau$ bankers who end date 2 with a portfolio $(1 + x_1(\theta_1), 0)$. With probability $(1-\tau)$ there are $(1-\alpha)(1-\theta_1)\lambda(1-\theta_2)(1-\tau)$ bankers who did not have to transfer cash at date 1 and receive no liquidity shock at date 2. This group of bankers holds unneeded cash and it is optimal for them to supply cash in the interbank market as long as $x_2(\theta_1, \theta_2) \geq R^{-1}$ (I will show later why this must hold). This group of bankers ends the period with a portfolio $(1 + x_1(\theta_1) + x_2(\theta_1, \theta_2), 0)$ or $(1 + x_1(\theta_1), 1)$. The expected portfolio of buyers without a shock at date 2 results to $(1 + x_1(\theta_1) + x_2(\theta_1, \theta_2)(1-\tau), 0)$ or $(1+x_1(\theta_1), 1-\tau)$, depending on $x_2(\theta_1, \theta_2)$.

The $\alpha(1-\theta_1)\theta_2$ illiquid bankers who receive a shock at date 2 need to exchange a fraction of their assets for cash. It is optimal for them to do so as long as $x_2(\theta_1, \theta_2) \leq 1$. However, since buyers hold $1+x_1(\theta_1)$ units of the asset, the price may be larger than one. The $\alpha(1-\theta_1)\theta_2$ illiquid bankers will end the period with a portfolio of $(\max[1 - x_2(\theta_1, \theta_2)], 0)$. The $\alpha(1-\theta_1)(1-\theta_2)$ illiquid bankers who do not receive a shock at either date have no gains from trade. They are assumed not to trade and end the period with a portfolio $(1, 0)$.

Finally, consider the hoarders. The $(1-\alpha)(1-\theta_1)(1-\lambda)\theta_2$ hoarders who receive a liquidity shock at date 2 use their own cash to discharge their debt and end the period with a portfolio $(1, 0)$. The alternative is to default. The $(1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2)$ hoarders who do not receive a liquidity shock can supply cash in the interbank market. It is optimal to supply cash as long as $x_2(\theta_1, \theta_2) \geq R^{-1}$ (again, I will show later why this must hold). This group of bankers end the period with a portfolio $(1, 1)$ or $(1 + x_2(\theta_1, \theta_2), 0)$, depending on $x_2(\theta_1, \theta_2)$. Figure 2 illustrates the allocations at date 0, 1 and 2.

**Date 3:** At date 3, bankers receive the realized returns of the portfolios they carry forward from date 2. Remember that cash has a return of one and assets have a return of $R > 1$ in units of cash. Bankers who have not discharged their debt so far, do so now.

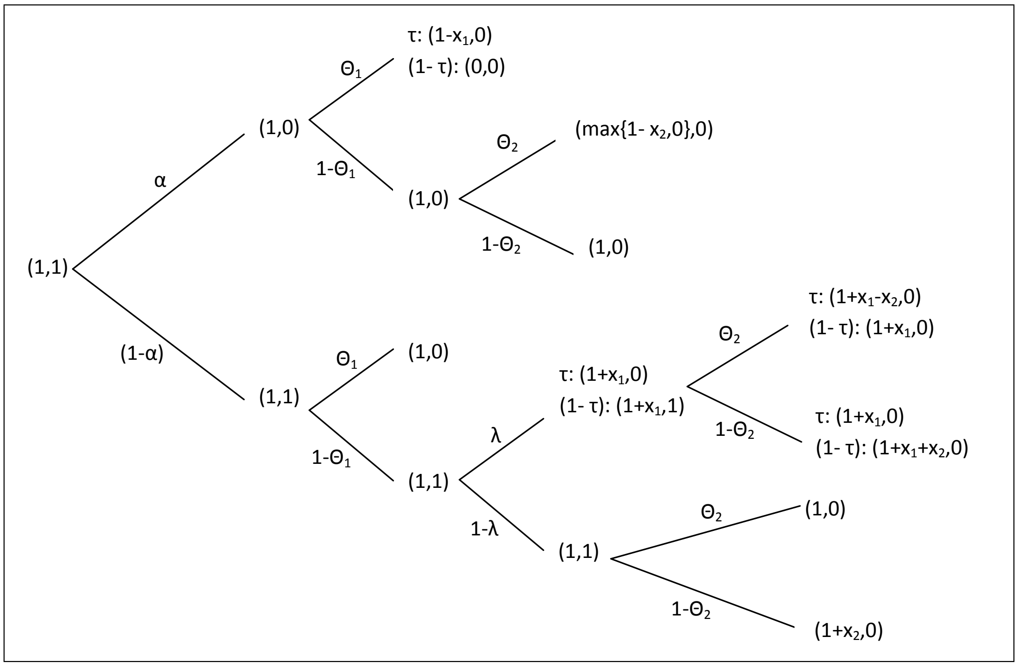


Figure 2: Allocations at date 0, 1 and 2

Then the economy ends. Figure 3 illustrates the final payoffs.

## 3.3 Market clearing

This section derives the market clearing prices $p_1(\theta_1)$ and $p_2(\theta_1, \theta_2)$. Since lotteries are ruled out at date 2, it holds that $p_2(\theta_1, \theta_2) = x_2(\theta_1, \theta_2)$. At date 1, trade with lotteries is possible and, hence, it holds that $p_1(\theta_1) = \frac{x_1(\theta_1)}{\tau}$. The model is solved backwards, beginning at date 2.

### 3.3.1 Market clearing at date 2

Although lotteries are ruled out at date 2, market clearing at date 2 will be affected by lotteries at date 1. Note that several groups of bankers are inactive at date 2 because they either demand no cash or cannot supply cash. First, every banker who received a shock in the previous period remains inactive at date 2. Second, hoarders who receive a shock at date 2 use their unit of cash to discharge their debt. Third, illiquid bankers without a shock at date 2 remain inactive since they only hold assets but no cash. Fourth, $\tau$ buyers without a shock at date 2 remain inactive because they only hold assets in their portfolio.

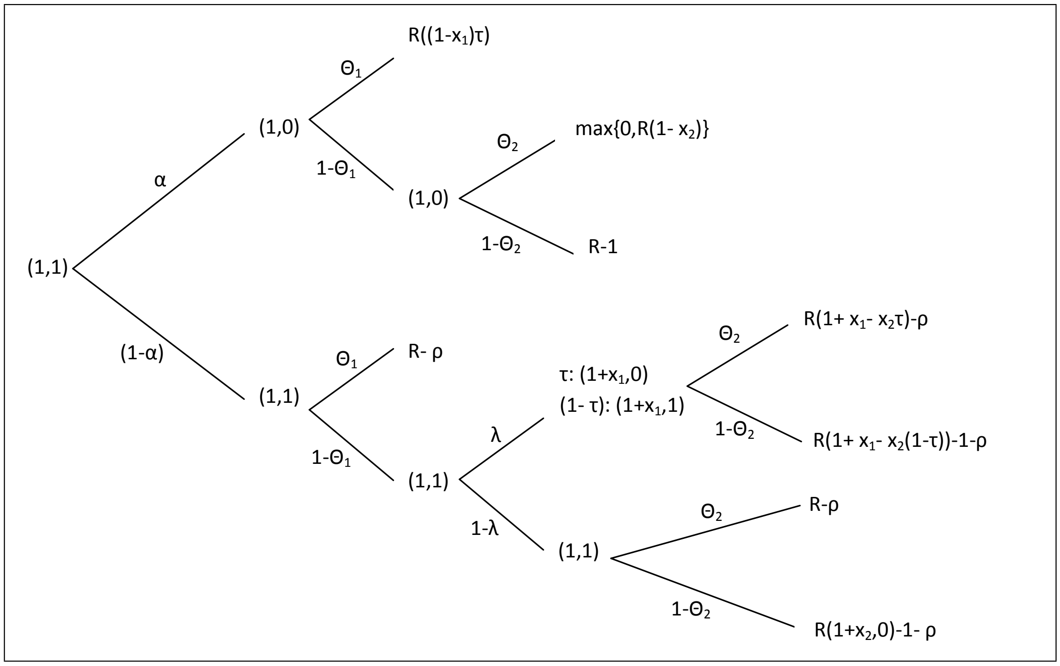


Figure 3: Final payoffs

Finally, the same is true for $(1-\tau)$ buyers with a shock at date 2.

The remaining four groups of bankers participate in the interbank market at date 2. The supply of cash comes from two groups of bankers. On the one hand, there is supply from hoarders without a shock at date 2. On the other hand, supply comes from $(1-\tau)$ buyers without a shock at date 2. Both groups will only supply cash if they are weakly better off compared to the realized payoff of their portfolios $(1,1)$ or $(1+x_1(\theta_1),1)$. These conditions can be stated as follows: For the hoarders it must hold that $R+1 \leq R\,[1+x_2(\theta_1,\theta_2)]$ which is $x_2(\theta_1,\theta_2) = p_2(\theta_1,\theta_2) \geq R^{-1}$, whereas for the $(1-\tau)$ buyers it must hold that $[1+x_1(\theta_1)]\,R+1 \leq R\,[1+x_1(\theta_1)+x_2(\theta_1,\theta_2)]$ or equivalently $[1+p_1(\theta_1)\tau]\,R+1 \leq R\,[1+p_1(\theta_1)\tau+p_2(\theta_1,\theta_2)]$ which again is $x_2(\theta_1,\theta_2) = p_2(\theta_1,\theta_2) \geq R^{-1}$. The supply is illustrated in Figure 4 and results to

$$(1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2)+(1-\alpha)(1-\theta_1)\lambda(1-\theta_2)(1-\tau). \tag{1}$$

The demand for cash comes from illiquid bankers and the $\tau$ buyers who receive a shock at date 2. The maximum demand from the illiquid bankers is

$$\alpha(1-\theta_1)\theta_2. \tag{2}$$

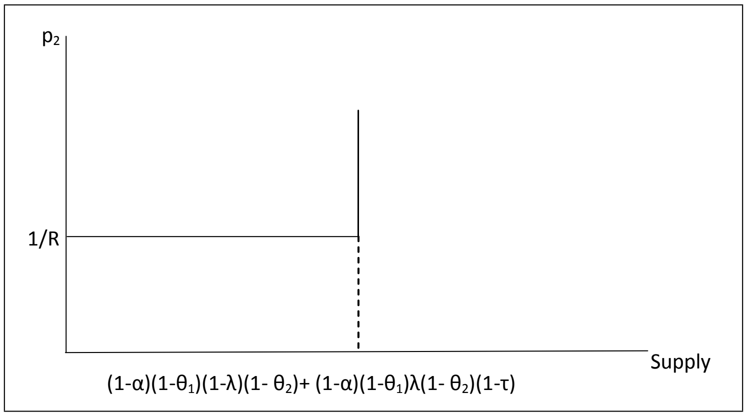


Figure 4: The supply of cash at date 2

Since this group holds a portfolio $(1, 0)$, these bankers can exchange a maximum of one unit of the asset in order to get cash. If $x_2(\theta_1, \theta_2) = p_2(\theta_1, \theta_2) < 1$ holds, it is optimal to offer all assets in order to get cash. If $x_2(\theta_1, \theta_2) = p_2(\theta_1, \theta_2) = 1$ holds, then these bankers are indifferent between obtaining cash and default. Hence, it is optimal to sell some of their assets.

The maximum demand from $\tau$ buyers who receive a shock at date 2 is

$$(1-\alpha)(1-\theta_1)\lambda\theta_2\tau. \tag{3}$$

If $x_2(\theta_1, \theta_2) < 1 + x_1(\theta_1)$, or equivalently $p_2(\theta_1, \theta_2) < 1 + p_1(\theta_1)\tau$ it is optimal to offer all assets in order to get cash. If $p_2(\theta_1, \theta_2) = 1 + p_1(\theta_1)\tau$, these bankers are indifferent between obtaining cash and default. Hence, it is optimal to sell some of their assets. The demand is illustrated in Figure 5.

From the supply and demand curves derived before, three different market clearing regimes are possible. They are defined by the three possible intersections of the demand and the supply curve in Figure 3 and 4. The first regime results if the supply of cash is so small that only some of the $\tau$ buyers can obtain cash. This is the case if $x_2(\theta_1, \theta_2) = 1 + x_1(\theta_1)$ or equivalently $p_2(\theta_1, \theta_2) = 1 + p_1(\theta_1)\tau$. For the first regime it must hold that

$$(1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2) + (1-\alpha)(1-\theta_1)\lambda(1-\theta_2)(1-\tau) < (1-\alpha)(1-\theta_1)\lambda\theta_2\tau,$$

which can be simplified to

$$\theta_2 > 1 - \lambda\tau.$$

Define $\theta_2^{**} = 1 - \lambda\tau$. The condition for the first regime can be denoted as $\theta_2^{**} < \theta_2$.

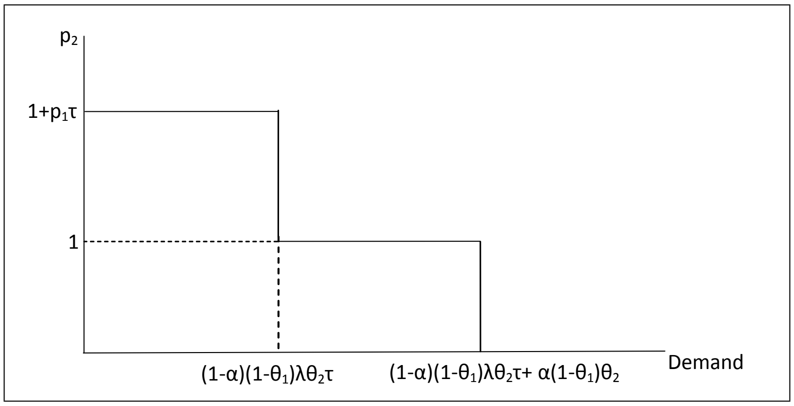


Figure 5: The demand for cash at date 2

The second regime results if the supply of cash is greater than the demand from the $\tau$ buyers and some, but not all, illiquid bankers. In this case, the market price will be $x_2(\theta_1, \theta_2) = p_2(\theta_1, \theta_2) = 1$. The second regime results if

$$\begin{aligned}(1-\alpha)(1-\theta_1)\lambda\theta_2\tau \quad < \quad & (1-\alpha)(1-\theta_1)(1-\lambda)(1-\theta_2) + (1-\alpha)(1-\theta_1)\lambda(1-\theta_2)(1-\tau) < \\ & (1-\alpha)(1-\theta_1)\lambda\theta_2\tau + \alpha(1-\theta_1)\theta_2.\end{aligned}$$

The inequality of the first two terms refers to the first regime and holds if $\theta_2^{**} > \theta_2$. The second part of the inequality can be rewritten as

$$(1-\alpha)(1-\lambda)(1-\theta_2) + (1-\alpha)\lambda(1-\theta_2)(1-\tau) < (1-\alpha)\lambda\theta_2\tau + \alpha\theta_2,$$

which can be simplified to

$$(1-\alpha)(1-\lambda\tau) < \theta_2.$$

Define $\theta_2^* = (1-\alpha)(1-\lambda\tau)$. Then the condition for the second regime can be denoted as $\theta_2^* < \theta_2$.

The third regime results if the supply is greater than the demand for cash. In this case, every banker who demands cash can obtain it. However, not every banker who is willing to supply cash is able to exchange it for assets. This implies that suppliers are indifferent between holding and selling cash. This is the case if $x_2(\theta_1, \theta_2) = p_2(\theta_1, \theta_2) = R^{-1}$. It is straightforward to see from the derivations before that the condition for the third regime can be denoted as $\theta_2^* > \theta_2$.

**Proposition 1** *The market clearing price at date 2 is denoted by $p_2(\theta_1, \theta_2)$ and defined* *by*

$$p_2(\theta_1, \theta_2) = \left\{ \begin{array}{c} R^{-1} \textit{ for } 0 \leq \theta_2 \leq \theta_2^* \\ 1 \textit{ for } \theta_2^* < \theta_2 < \theta_2^{**} \\ 1 + p_1(\theta_1)\tau \textit{ for } \theta_2^{**} < \theta_2 \leq 1 \end{array} \right\}, \quad (4)$$

*where* $\theta_2^* = (1 - \alpha)(1 - \lambda\tau)$ *and* $\theta_2^{**} = 1 - \lambda\tau$.

### 3.3.2 Market clearing at date 1

The characterization of market clearing at date 1 is more involved compared to the market clearing at date 2 since bankers need to take into account the prospective regarding date 2.[13] In order to derive the market clearing conditions, I proceed as follows. First, I show that there are always hoarders and buyers in equilibrium. Using this, I get a relation for the prices at date 1 and 2 which can be used to solve for the price $p_1(\theta_1)$. Then, I am able to show the inefficient hoarding if lotteries are ruled out. Finally, I introduce lotteries and prove that the inefficient hoarding is eliminated.

I start by showing that in equilibrium there are always hoarders and buyers. For this condition to hold, liquid bankers need to be indifferent between hoarding and buying. If the indifference condition is not fulfilled, bankers would strictly prefer one of the two alternatives. The investigation of these extreme cases allows to proof that there are always buyers and hoarders and, therefore, the indifference condition is fulfilled. The intuition for the proof is as follows.

Consider first the optimality of hoarding ($\lambda = 0$). If there are no buyers at date 1, illiquid bankers are indifferent between obtaining cash and default. In this case, the price at date 1 is at least one since trade with lotteries is possible. In addition, the price at date 2 must be smaller or equal to one since there are no buyers and only illiquid bankers who demand cash at date 2. The proof of Proposition 2 shows that it is optimal to hoard if $p_1(\theta_1) \leq E[p_2(\theta_1, \theta_2)]$. Since this is a contradiction, there are at least some buyers.

Consider next the optimality of buying ($\lambda = 1$). Note that in this case there is no need to use lotteries at date 1 ($\tau = 1$) because every demander can get cash. As a result, the price is smaller or equal to one. However, at date 2, the lack of supply implies $p_2(\theta_1, \theta_2) = 1 + p_1(\theta_1)\tau$ or equivalently $x_2(\theta_1, \theta_2) = 1 + x_1(\theta_1)$. The proof of Proposition 2 shows that it is optimal to hoard if $p_1(\theta_1) \geq E[p_2(\theta_1, \theta_2)]$ which is again a contradiction. As a result, there are at least some hoarders.

[13] This is the reason why the model is solved backwards.

**Proposition 2** *In equilibrium, $0 < \lambda(\theta_1) < 1$ must hold for every value of $\theta_1$. Thus, at date 1, liquid bankers are indifferent between hoarding cash and buying the asset. This holds if and only if*

$$p_1(\theta_1) = \int_0^1 p_2(\theta_1, \theta_2) f_2(\theta_2) d\theta_2. \tag{5}$$

**Proof.** See appendix. ■

Proposition 2 shows that liquid bankers without a shock at date 1 are indifferent between hoarding and buying. This condition is only fulfilled if and only if the following relation of prices at date 1 and 2 holds:

$$p_1(\theta_1) = E\left[p_2(\theta_1, \theta_2)|\theta_1\right].$$

Using the distribution of the random variable $p_2(\theta_1, \theta_2)$ derived in Proposition 1, the expected price at date 2 can be replaced by

$$p_2(\theta_1, \theta_2) = \left\{ \begin{array}{c} R^{-1} \text{ w. pr. } F_2\left\{(1-\alpha)\left[1-\lambda(\theta_1)\tau\right]\right\} \\ 1 \text{ w. pr. } F_2\left[1-\lambda(\theta_1)\tau\right] - F_2\left\{(1-\alpha)\left[1-\lambda(\theta_1)\tau\right]\right\} \\ 1 + p_1(\theta_1)\tau \text{ w. pr. } 1 - F_2\left[1-\lambda(\theta_1)\tau\right] \end{array} \right\},$$

which yields the following equation

$$p_1(\theta_1) = F_2\left\{(1-\alpha)\left[1-\lambda(\theta_1)\tau\right]\right\}(R^{-1}-1) - F_2\left[1-\lambda(\theta_1)\tau\right] p_1(\theta_1)\tau + 1 + p_1(\theta_1)\tau. \tag{6}$$

Solving for $p_1(\theta_1)$ yields the market price at date 1:

$$p_1(\theta_1) = \frac{1 - F_2\left\{(1-\alpha)\left[1-\lambda(\theta_1)\tau\right]\right\}(1-R^{-1})}{1 - \tau + F_2\left[1-\lambda(\theta_1)\tau\right]\tau}. \tag{7}$$

Note that $p_1(\theta_1)$ is a function of the endogenous variables $\lambda$ and $\tau$. Using the definition of prices, $p_1(\theta_1) = \frac{x_1(\theta_1)}{\tau}$, and applying it to Equation (6) gives

$$\begin{aligned} \frac{x_1(\theta_1)}{\tau} &= F_2\left\{(1-\alpha)\left[1-\lambda(\theta_1)\tau\right]\right\}(R^{-1}-1) - F_2\left[1-\lambda(\theta_1)\tau\right]x_1 + 1 + x_1 \qquad (8) \\ x_1(\theta_1) &= \frac{1 - F_2\left\{(1-\alpha)\left[1-\lambda(\theta_1)\tau\right]\right\}(1-R^{-1})}{\frac{1}{\tau} + F_2\left[1-\lambda(\theta_1)\tau\right] - 1}. \end{aligned}$$

Equation (7) can be expressed as a function of $\lambda \in (0,1)$. Moreover, if lotteries are ruled out at date 1 ($\tau = 1$), Equation (7) can be rewritten as

$$\widetilde{p}(\lambda) \equiv \frac{1 - F_2\left[(1-\alpha)(1-\lambda)\right](1-R^{-1})}{F_2(1-\lambda)}. \tag{9}$$

Note that the function $\widetilde{p}(\lambda)$ is increasing in $\lambda$ and varies from $1 - F_2\left[(1-\alpha)\right](1-R^{-1})$ to $\infty$ if $\lambda$ varies from 0 to 1. Consequently, there exists a unique value $\overline{\lambda} \in (0,1)$ which satisfies $\widetilde{p}(\lambda) = 1$ and it must hold that $\widetilde{p}(\overline{\lambda}) < 1$ if and only if $\lambda < \overline{\lambda}$. In what follows, I will show that there are two cases $\theta_1 < \overline{\theta_1}$ and $\theta_1 > \overline{\theta_1}$ where $\overline{\theta_1}$ is defined below. Assume for the moment that lotteries are ruled out, so $\tau = 1$ and, therefore, $p_1(\theta_1) = x_1(\theta_1)$.

**Case 1: $\theta_1 < \overline{\theta_1}$:**

If $\theta_1 < \overline{\theta_1}$, then $\widetilde{p}(\overline{\lambda}) < 1$ and the supply of cash is sufficient to provide cash to every demander. Thus, markets clear. Market clearing requires that the amount of bankers who demand cash is equal to the amount of bankers who supply cash. This is

$$(1-\alpha)(1-\theta_1)\lambda(\theta_1) = \alpha\theta_1$$

or equivalently

$$\lambda(\theta_1) = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}. \tag{10}$$

$\lambda(\theta_1)$ is chosen endogenously so that the supply and the demand of cash equalizes. Consider $\overline{\lambda}$ which results if $\widetilde{p}(\overline{\lambda}) = 1$. Let $\overline{\theta_1}$ be the unique value of $\theta_1$ that satisfies

$$\overline{\lambda} = \frac{\alpha\overline{\theta_1}}{(1-\alpha)(1-\overline{\theta_1})}.$$

The right hand side of the equation is increasing in $\theta_1$ and varies from 0 to $\infty$ as $\theta_1$ varies from 0 to 1. Consequently, there exists a unique solution to this equation and it satisfies $0 < \overline{\theta_1} < 1$. Note that if $\theta_1 < \overline{\theta_1}$, the following equation must hold

$$(1-\alpha)(1-\theta_1)\overline{\lambda} > \alpha\theta_1.$$

Hence, if markets clear, it must hold that $\lambda(\theta_1) < \overline{\lambda}$;

$$\lambda(\theta_1) = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)} < \overline{\lambda}.$$

**Case 2: $\theta_1 > \overline{\theta_1}$:**

If $\theta_1 > \overline{\theta_1}$, the supply of cash is insufficient to provide cash to every demander and, thus, markets do not clear. As a result, there are demanders who cannot obtain cash to discharge their debt because the supply of cash, or equivalently $\lambda(\theta_1)$, cannot increase above $\overline{\lambda}$. This can be represented as follows. If $\theta_1 \geq \overline{\theta_1}$, then

$$(1-\alpha)(1-\theta_1)\overline{\lambda} \leq \alpha\theta_1$$

and equilibrium requires $\lambda(\theta_1) = \overline{\lambda}$. Note that the above equation would only hold if $\lambda(\theta_1) > \overline{\lambda}$ and $\widetilde{p}(\lambda(\theta_1)) > 1$, but this is not possible so far. Hence,

$$p_1(\theta_1) = \min \left\{ \widetilde{p} \left[ \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)} \right], 1 \right\}.$$

The intuition for the above result is as follows: If $\theta_1$ increases and every demander can obtain cash to discharge his debt, the liquidity in the interbank market decreases. As a result, there is little cash left at date 2 and the price at date 2 must increase. Since the relation $p_1(\theta_1) = E[p_2(\theta_1, \theta_2)]$ must hold, the price at date 1 increases as well. At some point, the price at date 1 reaches its maximum value of $\widetilde{p}(\overline{\lambda}) = 1$. If $\theta_1$ would increase further and the demanders could still obtain cash at the price $p_1(\theta_1) = 1$, the price at date 2 would increase above one. This would violate the relation $p_1(\theta_1) = E[p_2(\theta_1, \theta_2)]$. Since this is not possible, the liquidity provision at date 1 has to stop and inefficient hoarding (rationing) occurs. In what follows, I will show that lotteries eliminate Case 2. So, there is no inefficient hoarding in equilibrium.

**Proposition 3** *The introduction of lotteries at date 1 always implies* $\lambda(\theta_1) = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}$ *and* $p_1(\theta_1) = \widetilde{p}\left[\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right]$. *Therefore, lotteries rule out inefficient hoarding for all* $\theta_1$.

In order to proof Proposition 3 assume for the moment that the introduction of lotteries at date 1 always implies market clearing. Thus, the following condition is always satisfied:

$$\lambda(\theta_1) = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}.$$

Replacing $\lambda(\theta_1)$ in Equation (8) yields

$$x_1(\theta_1) = \frac{1 - F_2\left\{(1-\alpha)\left[1 - \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\tau\right]\right\}(1-R^{-1})}{\frac{1}{\tau} + F_2\left[1 - \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\tau\right] - 1}. \tag{11}$$

**Proof. If $\theta_1 < \overline{\theta_1}$, then $\tau = 1$ and $x_1(\theta_1) < 1$:**

From Case 1 it is clear that $\theta_1 < \overline{\theta_1}$ induces $\lambda(\theta_1) < \overline{\lambda}$. This implies that $p_1(\theta_1) < 1$. Since $\tau = 1$, it must hold that $p_1(\theta_1) = x_1(\theta_1) < 1$. So,

$$x_1(\theta_1) = \frac{1 - F_2\left\{(1-\alpha)\left[1 - \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right]\right\}(1-R^{-1})}{F_2\left[1 - \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right]} < 1. \tag{12}$$

Thus, trade without lotteries takes place if $x_1(\theta_1) < 1$. In equilibrium, the following conditions must hold:

$$\lambda(\theta_1) = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}$$

and

$$p_1(\theta_1) = \widetilde{p}\left[\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right]$$

for every value of $\theta_1 < \overline{\theta_1}$.

**If** $\theta_1 > \overline{\theta_1}$**, then** $\tau \in (0,1)$ **and** $x_1 \leq 1$**:**

From Case 2 it is clear that $\theta_1 > \overline{\theta_1}$ induces $p_1(\theta_1) > 1$, which only is possible with lotteries $\tau \in (0,1)$. Consider Equation (11) and note that the only endogenous variable is $\tau$ since $\lambda$ is replaced by the market clearing condition. Hence, $x_1(\theta_1)$ is increasing in $\tau$ and varies from 0 to $\frac{1-F_2\left\{(1-\alpha)\left[1-\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right]\right\}(1-R^{-1})}{F_2\left[1-\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right]} < 1$ as $\tau$ varies from 0 to 1. Again, in equilibrium the following conditions must hold:

$$\lambda(\theta_1) = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}$$

and

$$p_1(\theta_1) = \widetilde{p}\left[\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right]$$

for every value of $\theta_1 \geq \overline{\theta_1}$. This completes the proof of Proposition 3. ■

The proof of Proposition 3 shows that $x_1(\theta_1)$ will never be larger than one although the liquidity shock may be larger than $\overline{\theta_1}$. If $\theta_1 \geq \overline{\theta_1}$, trade will take place using lotteries and if $\theta_1 < \overline{\theta_1}$, trade will take place without lotteries. As a result, in contrast to Case 2, the constraint induced by the asset holdings will never bind in trade with lotteries. Therefore, markets always clear and inefficient hoarding plays no role. Figure 6 illustrates $\tau$ as a function $\theta_1$ in equilibrium.

### 3.3.3 Market clearing at date 0

At date 0, a fraction $\alpha$ becomes illiquid bankers and a fraction $(1-\alpha)$ becomes liquid bankers. Related to the derivation of the indifference condition between buying and hoarding, I will show that bankers are indifferent between the two alternatives. If the indifference condition applies, $0 < \alpha < 1$ must hold in equilibrium. In order to derive the indifference condition, I will compare the expected payoff of a banker who becomes illiquid with the expected payoff of a hoarder. Note that I only have to take into account either the expected payoff of a hoarder or of a buyer since Proposition 2 showed that the expected payoffs of these alternatives are identical.

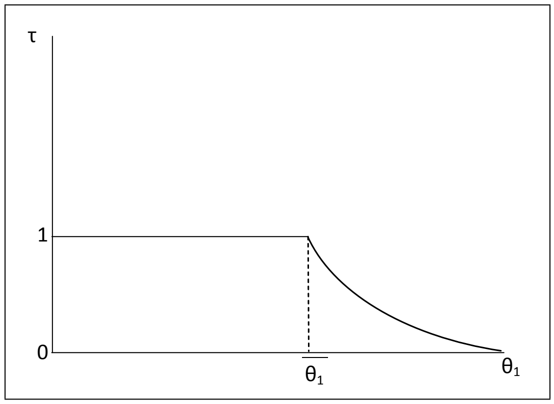


Figure 6: Equilibrium $\tau$ as a function of $\theta_1$

**Proposition 4** *If*

$$\int_0^1 \{(1-\theta_1)p_1(\theta_1)\tau + \theta_1\tau - \theta_1 - \theta_1\tau^2 p_1(\theta_1) + \\ (1-\theta_1)p_1(\theta_1)\tau\left[1 - F_2(\theta_2^{**})\right] E[\theta_2|\theta_2 > \theta_2^{**}]\} f_1(\theta_1)d\theta_1 = \frac{\rho}{R} \tag{13}$$

*holds, bankers are indifferent between holding cash and not holding cash at date 0. This implies that* $0 < \alpha < 1$ *must hold in equilibrium.*

**Proof.** See appendix. ■

## 3.4 Equilibrium

The equilibrium of the model is defined by the endogenous variables $\alpha$, $\lambda(\theta_1)$, $p_1(\theta_1)$, $p_2(\theta_2)$ and $\tau$ which satisfy the following conditions. Define $\widetilde{p}(\lambda)$ as

$$\widetilde{p}(\lambda) \equiv \frac{1 - F_2\left[(1-\alpha)(1-\lambda\tau)\right](1-R^{-1})}{1 - \tau + F_2\left(1-\lambda\tau\right)\tau}$$

for every $0 \leq \lambda \leq 1$. In equilibrium, the price at date 1 is

$$p_1(\theta_1) = \widetilde{p}\left[\frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}\right].$$

Since markets always clear, it must hold that

$$\lambda(\theta_1) = \frac{\alpha\theta_1}{(1-\alpha)(1-\theta_1)}.$$

In equilibrium, the price at date 2 is

$$p_2(\theta_1, \theta_2) = \left\{ \begin{array}{c} R^{-1} \text{ for } 0 \leq \theta_2 \leq \theta_2^* \\ 1 \text{ for } \theta_2^* < \theta_2 < \theta_2^{**} \\ 1 + p_1(\theta_1)\tau \text{ for } \theta_2^{**} < \theta_2 \leq 1 \end{array} \right\},$$

where $\theta_2^* = (1 - \alpha)(1 - \lambda\tau)$ and $\theta_2^{**} = 1 - \lambda\tau$.

Market clearing at date 0 requires indifference between holding cash and not holding cash at date 0. Thus, it must hold that

$$\int_0^1 \{(1 - \theta_1)p_1(\theta_1)\tau + \theta_1\tau - \theta_1 - \theta_1\tau^2 p_1(\theta_1) +$$
$$(1 - \theta_1)p_1(\theta_1)\tau \left[1 - F_2(\theta_2^{**})\right] E[\theta_2|\theta_2 > \theta_2^{**}]\} f_1(\theta_1)d\theta_1 = \frac{\rho}{R}.$$

Finally, $\tau \in (0, 1)$ if $\theta_1 > \overline{\theta_1}$ and $\tau = 1$ if $\theta_1 < \overline{\theta_1}$.

# 4 Portfolio choice

The second inefficiency of insufficient liquidity holdings at date 0 (portfolio choice) has not been considered so far but is the purpose of this section. I will follow the approach of Gale and Yorulmazer (2011) who set up a simplified version of the model that consists of three dates $t = 0, 1, 2$. Identical to the basic model, bankers are initially endowed with a portfolio $(1, 1)$ and decide whether to become illiquid or remain liquid at date 0. At date 1, bankers may receive a liquidity shock and can trade assets for cash in an interbank market. At date 2, returns are realized and the outstanding debt is discharged. Again, I will introduce lotteries and compare the equilibrium outcome to the model without lotteries and to the planner's solution. Since trade takes place only at date 1, there is no role for hoarding.

The model is solved backwards, starting at date 1. At the beginning of date 1, $\theta$ bankers receive a liquidity shock. The $(1 - \alpha)\theta$ liquid bankers can discharge their debt using their own cash holdings and end the period with a portfolio $(1, 0)$ and a return of $R - \rho$. The $\alpha(1 - \theta)$ illiquid bankers receive no shock and end the period with a portfolio $(1, 0)$ and a return of $R - 1$. The demand for cash comes from the $\alpha\theta$ bankers who need to exchange assets for cash in order to discharge their debt. The supply of cash comes from the $(1 - \alpha)(1 - \theta)$ liquid bankers without shock.

Market clearing, again, depends on $\theta$. If the shock $\theta$ is small, then the supply of cash is larger than the demand. In this case, there is no need to offer lotteries ($\tau = 1$) and the suppliers of cash need to be indifferent between trading or not. Thus, the price is

$p(\theta) = x(\theta) = R^{-1}$. If the demand for cash is larger than the supply, bankers need to be indifferent between obtaining cash and default. This implies that bankers transfer all their asset holdings $x(\theta) = 1$. Moreover, I assume that $\tau$ is approximately zero (this assumption is valid since the indifference condition is still fulfilled). Therefore, the price $p(\theta) = \frac{1}{\tau}$ goes to infinity. The market price is defined by

$$p(\theta) = \left\{ \begin{array}{c} R^{-1} \text{ if } \theta < 1 - \alpha \\ \infty \text{ if } \theta > 1 - \alpha \end{array} \right\}. \tag{14}$$

The expected payoff of the illiquid bankers who offer lotteries is as follows. With probability $\tau$ they receive cash and can discharge their debt. Hence, they end date 1 with a portfolio $(1 - x(\theta), 0)$ and their return results to $R[1 - x(\theta)]$. With probability $(1 - \tau)$, they will not receive cash and cannot discharge their debt. Consequently, they are forced to default and end the period with a portfolio $(0, 0)$ and a return of 0. The expected payoff of the liquid bankers is as follows. With probability $\tau$ they need to transfer cash and end date 1 with a portfolio $(1 + x(\theta), 0)$ and a return of $R[1 + x(\theta)] - 1$. With probability $(1 - \tau)$, they transfer no cash and end date 1 with a portfolio $(1 + x(\theta), 1)$. Hence, their return is $R[1 + x(\theta)]$.

The illiquid banker's expected payoff satisfies

$$\int_0^1 \{\theta [1 - x(\theta)] \tau R + (1 - \theta)(R - 1)\} f(\theta) d\theta.$$

The liquid banker's expected payoff satisfies

$$\int_0^1 [\theta R + (1 - \theta) \{[1 + x(\theta)] R - 1\} \tau + (1 - \theta) [1 + x(\theta)] R(1 - \tau)] f(\theta) d\theta - \rho.$$

Using the fact that $\tau = 1$ and $p(\theta) = x(\theta) = R^{-1}$ if $\theta_1 < 1 - \alpha$ and $\tau$ which is approximately zero and $x(\theta) = 1$ if $\theta_1 > 1 - \alpha$, I can break up the integral and rewrite the expected payoff of illiquid bankers as follows

$$\int_0^{1-\alpha} \left[\theta(1 - \frac{1}{R})R + (1 - \theta)(R - 1)\right] f(\theta) d\theta + \int_{1-\alpha}^1 [(1 - \theta)(R - 1)] f(\theta) d\theta. \tag{15}$$

Analogously, the expected payoff of liquid bankers can be rewritten as follows

$$\int_0^{1-\alpha} \left\{\theta R + (1 - \theta) \left[(1 + \frac{1}{R})R - 1\right]\right\} f(\theta) d\theta + \int_{1-\alpha}^1 [\theta R + (1 - \theta) 2R] f(\theta) d\theta - \rho. \tag{16}$$

In equilibrium the expected payoff of illiquid bankers has to be equal to the expected payoff of a liquid banker. Note that several terms cancel if I set Equation (15) equal to

Equation (16). This yields

$$\rho + \int_0^{1-\alpha} (-1) f(\theta) d\theta = \int_{1-\alpha}^{1} [2R - \theta R - (1-\theta)(R-1)] f(\theta) d\theta,$$

which is

$$\rho - \int_0^{1-\alpha} (1) f(\theta) d\theta = \int_{1-\alpha}^{1} (R + 1 - \theta) f(\theta) d\theta$$

or

$$\rho - \int_0^{1-\alpha} (1) f(\theta) d\theta = \int_{1-\alpha}^{1} (R) f(\theta) d\theta + \int_{1-\alpha}^{1} (1) f(\theta) d\theta - \int_{1-\alpha}^{1} (\theta) f(\theta) d\theta. \tag{17}$$

In the basics of the model I specified that the random variables $\theta_1$ and $\theta_2$ are iid with support $[0,1]$. Furthermore, the random variables have a density function $f(\theta_i)$ and a cumulative density function $F(\theta_i)$. However, no distributional assumption regarding the random variables was made. In order to solve Equation (17) I assume that the random variables follow a standard uniform distribution. As a result, it holds that $f(\theta) = 1$ and $F(\theta) = \theta$;

$$\rho - (1-\alpha) = R - R(1-\alpha) + 1 - (1-\alpha) - \left[ \frac{1^2}{2} - \frac{(1-\alpha)^2}{2} \right].$$

Simplifying the above equation results to

$$\rho - \frac{1}{2} = R\alpha + \frac{(1-\alpha)^2}{2}$$

and rearranging yields

$$0 = \frac{1}{2}\alpha^2 + (R-1)\alpha + (1-\rho).$$

This is a quadratic equation in $\alpha$ and can be solved with the quadratic formula. The resulting two solutions are as follows:

$$\alpha_{1,2} = (1-R) \pm \sqrt{(R-1)^2 - 2(1-p)}.$$

Note that I can rule out the solution $\alpha = (1-R) - \sqrt{(R-1)^2 + 2(p-1)}$ since this implies a negative value for $\alpha$ which is not possible. Hence, in equilibrium, the measure of bankers who choose to remain liquid $(1-\alpha)$ is determined by

$$1 - \alpha = R - \sqrt{(R-1)^2 - 2(1-p)}.$$

Now, the portfolio choice can be compared to the equilibrium outcome in the model without lotteries and to the planner's solution. First, I will consider the comparison of the model with lotteries and without lotteries and then I compare the portfolio choice of the social planner and the model with lotteries.

Gale and Yorulmazer (2011) analyze the initial portfolio choice in the laisser-faire equilibrium and compare it with the constrained efficient allocation of a social planner. They find that the portfolio choice at date 0 is not constrained optimal since the value of cash provided to the creditors is not internalized. Identical to the model with lotteries, at date 0, bankers have to be indifferent between being liquid and illiquid. The corresponding payoffs are the same as before, however, remember that $\tau = 1$ and therefore, $x(\theta) = p(\theta)$. Markets will clear at a price defined by

$$p(\theta) = \left\{ \begin{array}{c} R^{-1} \text{ if } \theta_1 < 1 - \alpha \\ 1 \text{ if } \theta_1 > 1 - \alpha \end{array} \right\}. \tag{18}$$

Setting up the indifference condition and simplifying results to

$$E[p(\theta)] = \frac{\rho}{R}.$$

Finally, replacing $E[p(\theta)]$ with the expected market price from Equation (18) leads to

$$\begin{aligned} (1 - F_1(1-\alpha)) + F_1(1-\alpha)\frac{1}{R} &= \frac{\rho}{R} \\ F_1(1-\alpha) &= \frac{R-\rho}{R-1}. \end{aligned}$$

Under a standard normal distribution the amount of bankers who choose to remain liquid in the model without lotteries amounts to $\frac{R-\rho}{R-1}$.

Now, the portfolio choice of the model with and without lotteries can be compared. In order to do so, I use a graphical approach and plot the difference of the two expressions with the parameters $\rho$ and $R$ varying on the interval $[1, 3]$ (remember that $\rho, R > 1$) in Figure 7.[14] I find a positive difference which implies that there are more bankers who decide to remain liquid at date 0 if lotteries are introduced. Concerning the robustness of this result, I tried various other intervals and always came to the same conclusion. As a result, the introduction of lotteries can at least mitigate the inefficient portfolio choice identified by Gale and Yorulmazer (2011).

In order to determine the optimal amount of cash accumulation $m_0$, the social planner takes the marginal cost $\rho > 1$ and the marginal value of cash into account. The marginal value of cash is 1 if $\theta_1 < m_0$ and $R + 1$ if $\theta_1 > m_0$. The intuition for this is as follows: If

[14] I used Mathematica 7.0 with the command `Plot3D[(R - ((R - 1)^2 - 2 (1 - p))^0.5 - (R - p)/(R - 1)), {R, 1, 3}, {p, 1, 3}]`.

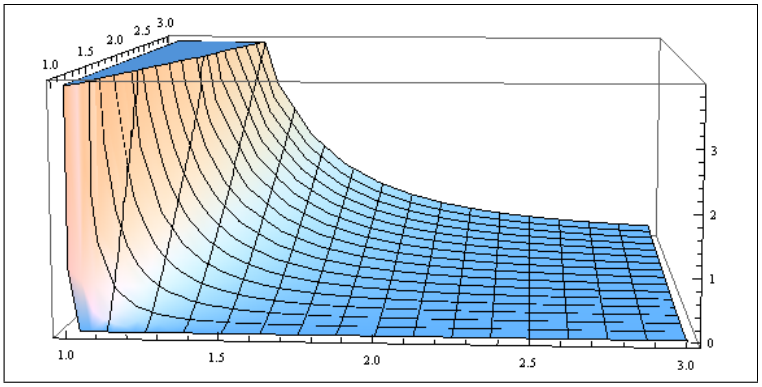


Figure 7: Portfolio choice - with vs. without lotteries

the fraction of illiquid bankers who demand cash is smaller than $m_0$, then every illiquid banker in need of cash can obtain cash. In this case, an additional unit of cash yields a return of one. However, if $m_0$ is insufficient to provide cash to every banker in need, the value of an additional unit of cash is equal to $1 + R$ because it saves an asset with a return of $R$ in addition. The planner's first order condition equates the marginal value and the marginal cost of cash. This is

$$R\left[1 - F_1(m_0)\right] + 1 = \rho.$$

Rearranging yields

$$F_1(m_0) = \frac{R + 1 - \rho}{R}.$$

Again, under a standard normal distribution, the amount of cash a social planner accumulates is $\frac{R+1-\rho}{R}$. As before, I compare the measure of bankers who remain liquid in the model with lotteries to the amount of cash the social planner accumulates. Again, I use a graphical approach and plot the difference between the two in Figure 8 and 9. Here, a positive difference would imply that the model with lotteries implies a larger level of liquidity holdings than the social planner would choose.

Figures 8 and 9 show that I can find parameter values where the difference is either positive or negative.[15] Thus, for certain parameter values the model with lotteries implies a higher level of liquidity in the interbank market than the social planner would choose. Although this result might seem puzzling, there is a possible explanation for this. The planner optimizes his program without using lotteries and, therefore, under the restric-

[15] Again, I used Mathematica 7.0 with the command: `Plot3D[(R - ((R - 1)^2 - 2 (1 - p))^0.5 - (R + 1 - p)/R), {R, 1, 3}, {p, 1, 3}]` for Figure 8 and `Plot3D[(R - ((R - 1)^2 - 2 (1 - p))^0.5 - (R + 1 - p)/R), {R, 1, 10}, {p, 1, 10}]` for Figure 9.

tion of indivisible cash. If the social planner would be able to use lotteries, the above comparison would probably lead to a different conclusion. Another interpretation of the above results is that the model with lotteries induces an equilibrium where too many bankers are holding cash in their portfolio.

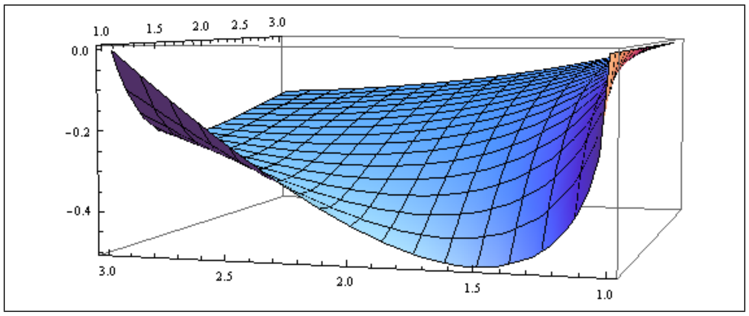


Figure 8: Portfolio choice - with lotteries vs. social planner

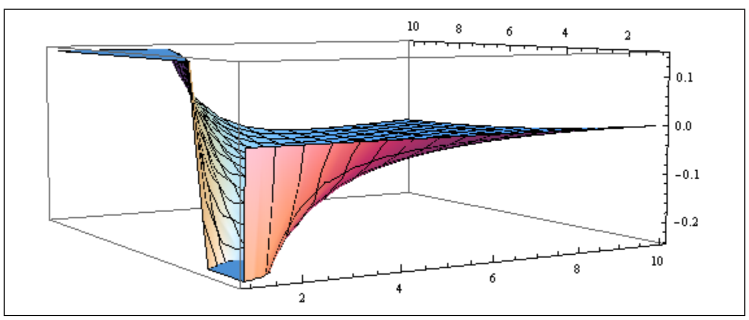


Figure 9: Portfolio choice - with lotteries vs. social planner (2)

To sum up, under the assumption of a standard uniform distribution for the random variables, I can conclude that the introduction of lotteries can at least mitigate the inefficient portfolio choice identified in the basic model by Gale and Yorulmazer (2011).

# 5 Conclusion

This paper introduced randomized trading schemes into the interbank market model developed by Gale and Yorulmazer (2011). The motivation was to show whether the introduction of lotteries can mitigate or even eliminate the inefficiencies Gale and Yorulmazer (2011) identify in their model. I was able to show that lotteries over cash induces market clearing at any time since prices in the interbank market at date 1 are not restricted to be smaller or equal to one. As a result, the introduction of lotteries eliminates inefficient hoarding at date 1. Under some distributional assumptions, the second inefficiency identified by Gale and Yorulmazer (2011) can be mitigated. The comparison of the portfolio choice in the model with lotteries compared to the model by Gale and Yorulmazer (2011) shows that the introduction of lotteries over cash induces more bankers to remain liquid and, therefore, implies a higher level of liquidity in the interbank market. The comparison of the portfolio choice in the model with lotteries and the social planner's choice suggests that for some parameter values, the model with lotteries implies larger liquidity holdings than the social planner would choose. Although this result might seem puzzling, there is a possible explanation for this. Since the planner is not able to make use of lotteries, he optimizes his program under the constraint of the indivisible character of cash. An interesting extension to pursue would be to introduce lotteries at date 2 and analyze market clearing. In particular, it would be interesting to study the effects on the initial portfolio choice. Finally, the findings of this paper suggest little robustness of Gale and Yorulmazer (2011) regarding the indivisible character of cash.

# A Appendix

## A.1 Proofs

**Proof of Proposition 2** Buyers end date 1 with a portfolio $(1 + p_1(\theta_1)\tau, 1 - \tau)$. A fraction $\theta_2$ receives a liquidity shock and end date 2 with an expected payoff $(1 + p_1(\theta_1)\tau - p_2(\theta_1, \theta_2)\tau)R$. A fraction $(1 - \theta_2)$ receives no shock and has the expected payoff $(1 + p_1(\theta_1)\tau + (1 - \tau)p_2(\theta_1, \theta_2))R - 1$. As a result, the buyers' expected payoff at date 1 results to

$$\begin{aligned} & \int_0^1 (\theta_2 \left\{ [1 + p_1(\theta_1)\tau - p_2(\theta_1, \theta_2)\tau] R \right\} + \\ & (1 - \theta_2) \left\{ [1 + p_1(\theta_1)\tau + p_2(\theta_1, \theta_2)(1 - \tau)] R - 1 \right\}) f_2(\theta_2) d\theta_2 \\ = & \int_0^1 \left\{ [1 + p_1(\theta_1)\tau + (1 - \tau - \theta_2)p_2(\theta_1, \theta_2)] R - (1 - \theta_2) \right\} f_2(\theta_2) d\theta_2. \end{aligned}$$

Hoarders end date 1 with a portfolio $(1, 1)$. A fraction $\theta_2$ of hoarders receives a liquidity shock and has a payoff $R$. A fraction $(1 - \theta_2)$ of the hoarders receives no shock and ends date 2 with an expected payoff $(1 + p_2(\theta_1, \theta_2))R - 1$. The hoarders' expected payoff at date 1 is identical to Gale and Yorulmazer (2011) and results to

$$\begin{aligned} & \int_0^1 (\theta_2 R + (1 - \theta_2) \left\{ [1 + p_2(\theta_1, \theta_2)] R - 1 \right\}) f_2(\theta_2) d\theta_2 \\ = & \int_0^1 \left\{ [1 + (1 - \theta_2)p_2(\theta_1, \theta_2)] R - (1 - \theta_2) \right\} f_2(\theta_2) d\theta_2. \end{aligned}$$

It is optimal to buy if and only if the buyers' payoff is at least as great as the hoarders', that is,

$$\begin{aligned} & \int_0^1 \left\{ [1 + p_1(\theta_1)\tau + (1 - \tau - \theta_2)p_2(\theta_1, \theta_2)] R - (1 - \theta_2) \right\} f_2(\theta_2) d\theta_2 \\ \geq & \int_0^1 \left\{ [1 + (1 - \theta_2)p_2(\theta_1, \theta_2)] R - (1 - \theta_2) \right\} f_2(\theta_2) d\theta_2, \end{aligned}$$

which is

$$p_1(\theta_1) \geq \int_0^1 p_2(\theta_1, \theta_2) f_2(\theta_2) d\theta_2.$$

It is optimal to hoard if and only if

$$p_1(\theta_1) \leq \int_0^1 p_2(\theta_1, \theta_2) f_2(\theta_2) d\theta_2.$$

Proposition 1 derived the distribution of the random variable $p_2(\theta_1, \theta_2)$:

$$p_2(\theta_1,\theta_2) = \left\{ \begin{array}{c} R^{-1} \text{ w. pr. } F_2\left\{(1-\alpha)\left[1-\lambda(\theta_1)\tau\right]\right\} \\ 1 \text{ w. pr. } F_2\left[1-\lambda(\theta_1)\tau\right] - F_2\left\{(1-\alpha)\left[1-\lambda(\theta_1)\tau\right]\right\} \\ 1+p_1(\theta_1)\tau \text{ w. pr. } 1-F_2\left[1-\lambda(\theta_1)\tau\right] \end{array} \right\}.$$

The expected value of $p_2(\theta_1, \theta_2)$ is

$$\begin{aligned} E[p_2(\theta_1,\theta_2)|\theta_1] &= F_2\left\{(1-\alpha)\left[1-\lambda(\theta_1)\tau\right]\right\}R^{-1} + F_2\left[1-\lambda(\theta_1)\tau\right] - \\ &\quad F_2\left\{(1-\alpha)\left[1-\lambda(\theta_1)\tau\right]\right\} + \left\{1-F_2\left[1-\lambda(\theta_1)\tau\right]\right\}\left[1+p_1(\theta_1)\tau\right] \\ &= F_2\left\{(1-\alpha)\left[1-\lambda(\theta_1)\tau\right]\right\}(R^{-1}-1) - F_2\left[1-\lambda(\theta_1)\tau\right]p_1(\theta_1)\tau + 1 + p_1(\theta_1)\tau. \end{aligned}$$

Consider the case of optimality of hoarding $\lambda(\theta_1) = 0$. This implies that illiquid bankers are indifferent between obtaining cash and default, which means $p_1(\theta_1) = \frac{x_1(\theta_1)}{\tau} \geq 1$. Setting $\lambda(\theta_1) = 0$, the expected value of $p_2(\theta_1, \theta_2)$ is

$$E[p_2(\theta_1,\theta_2)|\theta_1] = F_2(1-\alpha)(R^{-1}-1) + 1 < 1.$$

But optimality of hoarding at date 1 requires

$$p_1(\theta_1) \leq \int_0^1 p_2(\theta_1,\theta_2) f_2(\theta_2) d\theta_2.$$

This is a contradiction and proves that $\lambda(\theta_1) > 0$. Thus, there will be at least some buyers.

Consider next the case of optimality of buying $\lambda(\theta_1) = 1$. Due to the absence of hoarding, there is no need to trade using lotteries ($\tau = 1$). Note that if $\lambda(\theta_1) = \tau = 1$, there is no supply of cash at date 2 and the market clearing price requires that

$$E[p_2(\theta_1,\theta_2)] = 1 + p_1(\theta_1).$$

However, optimality of buying at date 1 requires

$$p_1(\theta_1) \geq \int_0^1 p_2(\theta_1,\theta_2) f_2(\theta_2) d\theta_2.$$

Again, this is a contradiction and proves that $\lambda(\theta_1) < 1$. So, there will be at least some hoarders.

As a consequence, bankers must be indifferent between hoarding and buying and therefore, $0 < \lambda(\theta_1) < 1$. Hence,

$$p_1(\theta_1) = \int_0^1 p_2(\theta_1, \theta_2) f_2(\theta_2) d\theta_2.$$

**Proof of Proposition 4** The expected return of a banker who chooses to remain liquid and to become a hoarder is as follows. With probability $\theta_1$, he receives a liquidity shock at $t = 1$ and discharges his debt using the cash he holds in his portfolio. His return is $R$. With probability $(1-\theta_1)\theta_2$, he receives no shock at $t = 1$ but at $t = 2$. In this case, his return is again $R$. With probability $(1-\theta_1)(1-\theta_2)$, he will not receive a shock at either date and uses his cash to acquire $p_2(\theta_1, \theta_2)$ units of the asset at $t = 2$. His return is $(1 + p_2(\theta_1, \theta_2))R - 1$. Hence, the expected return can be written as

$$\int_0^1 \int_0^1 (\theta_1 R + (1-\theta_1)\theta_2 R + (1-\theta_1)(1-\theta_2)\left\{[1 + p_2(\theta_1, \theta_2)]\, R - 1\right\}) f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2 - \rho.$$

Rearranging this equation yields

$$R + \int_0^1 \int_0^1 \left\{(1-\theta_1)(1-\theta_2)\left[p_2(\theta_1, \theta_2)R - 1\right]\right\} f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2 - \rho.$$

Using the relation of the price at date 1 and the expected price at date 2 yields

$$R + \int_0^1 (1-\theta_1) R \underbrace{\left[\int_0^1 p_2(\theta_1, \theta_2) f_2(\theta_2) d\theta_2\right]}_{=\ p_1(\theta_1)} f_1(\theta_1) d\theta_1 -$$
$$\int_0^1 \int_0^1 (1-\theta_1)\left[\theta_2 p_2(\theta_1, \theta_2) R + (1-\theta_2)\right] f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2 - \rho.$$

Again, rearranging yields

$$R + \int_0^1 \left[(1-\theta_1) p_1(\theta_1) R\right] f_1(\theta_1) d\theta_1 -$$
$$\int_0^1 \int_0^1 \left\{(1-\theta_1)\left[\theta_2 p_2(\theta_1, \theta_2) R + (1-\theta_2)\right]\right\} f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2 - \rho.$$

As a result, a banker who chooses to become a hoarder will get a return of $R$ for sure. If the banker can buy assets in exchange for cash, the return might even be larger.

The expected return of a banker who chooses to become illiquid can be calculated as follows. With probability $(1-\theta_1)(1-\theta_2)$, he will not receive a liquidity shock at either date and his return is $R - 1$. With probability $\theta_1$, he receives a shock at $t = 1$. In this case he needs to sell a fraction of his assets in order to obtain cash. Otherwise he would default. The resulting return is $(1-p_1(\theta_1)\tau)\tau R$. With probability $(1-\theta_1)\theta_2$, the banker receives a shock at date 2 but not at date 1. Again, the banker needs to exchange assets for cash. His return is $\max\{0,(1-p_2(\theta_1,\theta_2))R\}$. Hence, the expected return can be written as:

$$\int_0^1\int_0^1 (\theta_1\left[1-p_1(\theta_1)\tau\right]\tau R+(1-\theta_1)(1-\theta_2)(R-1)+$$
$$(1-\theta_1)\theta_2\max\{0,\left[1-p_2(\theta_1,\theta_2)\right]R\})f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2.$$

Rearranging this equation yields

$$R+\int_0^1\left[\theta_1\tau R-\theta_1 R-\theta_1\tau^2 R p_1(\theta_1)\right]f_1(\theta_1)d\theta_1-$$
$$\int_0^1\int_0^1\{(1-\theta_1)\left[\theta_2 p_2(\theta_1,\theta_2)R+(1-\theta_2)\right]\}f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2+$$
$$\int_0^1\int_{\theta_2>\theta_2^{**}}^1\left\{(1-\theta_1)\theta_2\underbrace{\left[1-p_2(\theta_1,\theta_2)\right]}_{=\,-p_1(\theta_1)\tau}R\right\}f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2.$$

Again, using the relation of the price at date 1 and the expected price at date 2 yields

$$R+\int_0^1\left[\theta_1\tau R-\theta_1 R-\theta_1\tau^2 R p_1(\theta_1)\right]f_1(\theta_1)d\theta_1-$$
$$\int_0^1\int_0^1\{(1-\theta_1)\left[\theta_2 p_2(\theta_1,\theta_2)R+(1-\theta_2)\right]\}f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2-$$
$$\int_0^1\int_{\theta_2>\theta_2^{**}}^1\left[(1-\theta_1)\theta_2 p_1(\theta_1)\tau R\right]f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2,$$

since $1-p_2(\theta_1,\theta_2)=-p_1(\theta_1)\tau$ for $\theta_2>\theta_2^{**}$.

If in equilibrium $0<\alpha<1$ holds, illiquid bankers and hoarders need to have the same expected return. Notice that the second term in both expected returns is identical and cancels out.

$$\int_0^1\left[(1-\theta_1)p_1(\theta_1)\tau R\right]f_1(\theta_1)d\theta_1-\rho=$$
$$\int_0^1\left[\theta_1\tau R-\theta_1 R-\theta_1\tau^2 R p_1(\theta_1)\right]f_1(\theta_1)d\theta_1-$$
$$\int_0^1\int_{\theta_2>\theta_2^{**}}^1\left[(1-\theta_1)\theta_2 p_1(\theta_1)\tau R\right]f_1(\theta_1)f_2(\theta_2)d\theta_1 d\theta_2.$$

Rearranging this equation yields

$$
\begin{aligned}
&\int_0^1 \left[(1-\theta_1)p_1(\theta_1)\tau\right] f_1(\theta_1)d\theta_1 - \frac{\rho}{R} = \\
&\int_0^1 \left[\theta_1\tau - \theta_1 - \theta_1\tau^2 p_1(\theta_1)\right] f_1(\theta_1)d\theta_1 - \\
&\int_0^1 \int_{\theta_2>\theta_2^{**}}^1 \left[(1-\theta_1)\theta_2 p_1(\theta_1)\tau\right] f_1(\theta_1) f_2(\theta_2) d\theta_1 d\theta_2.
\end{aligned}
$$

Using the identity $\int_{\theta_2>\theta_2^{**}}^1 \theta_2 f_2(\theta_2)d\theta_2 = (1-F_2(\theta_2^{**}))E[\theta_2|\theta_2 > \theta_2^{**}]$ allows the rewrite the equation as

$$
\begin{aligned}
&\int_0^1 \left[(1-\theta_1)p_1(\theta_1)\tau\right] f_1(\theta_1)d\theta_1 - \frac{\rho}{R} = \\
&\int_0^1 \left[\theta_1\tau - \theta_1 - \theta_1\tau^2 p_1(\theta_1)\right] f_1(\theta_1)d\theta_1 + \\
&\int_0^1 \left\{ (1-\theta_1)p_1(\theta_1)\tau \underbrace{\left[\int_{\theta_2>\theta_2^{**}}^1 \theta_2 f_2(\theta_2)d\theta_2\right]}_{=\ (1-F_2(\theta_2^{**}))E[\theta_2|\theta_2>\theta_2^{**}]} \right\} f_1(\theta_1)d\theta_1.
\end{aligned}
$$

Finally, the indifference condition can be stated as follows

$$
\begin{aligned}
&\int_0^1 \{(1-\theta_1)p_1(\theta_1)\tau + \theta_1\tau - \theta_1 - \theta_1\tau^2 p_1(\theta_1) + \\
&(1-\theta_1)p_1(\theta_1)\tau \left[1-F_2(\theta_2^{**})\right] E[\theta_2|\theta_2 > \theta_2^{**}]\} f_1(\theta_1)d\theta_1 = \frac{\rho}{R}.
\end{aligned}
$$

## A.2 Eidesstattliche Erklärung

Ich bezeuge mit meiner Unterschrift, dass meine Angaben über die bei der Abfassung meiner Arbeit benützten Hilfsmittel sowie über die mir zuteil gewordene Hilfe in jeder Hinsicht der Wahrheit entsprechen und vollständig sind. Ich habe das Merkblatt zu Plagiat und Betrug vom 23.11.05 gelesen und bin mir der Konsequenzen eines solchen Handelns bewusst.

Benjamin Müller
Basel, 8. Januar 2012

# References


[1] Acharya, Viral, and Ouarda Merrouche (2009). "Precautionary Hoarding of Liquidity and Interbank Markets: Evidence from Subprime Crisis," Working Paper, New York University Stern School of Business.

[2] Acharya, Viral, Hyun Song Shin, and Tanju Yorulmazer (2010). "Crisis Resolution and Bank Liquidity," *Review of Financial Studies*, 24(6), 2166-2205.

[3] Afonso, Gara, Anna Kovner, and Antoinette Schoar (2011). "Stressed, not Frozen: The Federal Funds Market in the Financial Crisis," *Journal of Finance,* 66(4), 1109-1139.

[4] Allen, Franklin, and Douglas Gale (2004). "Financial Intermediaries and Markets," *Econometrica*, 72, 1023-1061.

[5] Allen, Franklin, and Douglas Gale (2005). "From Cash-in-the-Market Pricing to Financial Fragility," *Journal of the European Economic Association*, 3(2-3), 535-546.

[6] Bhattacharya, Sudipto, and Douglas Gale (1987). "Preference Shocks, Liquidity and Central Bank Policy," in W. Barnett and K. Singleton, eds., New Approaches to Monetary Economics, Cambridge University Press, 69-88.

[7] Berentsen, Aleksander, Samuel Huber, and Alessandro Marchesiani (2011). "Free-riding on Liquidity," ECON - Working Paper, Department of Economics - University of Zurich.

[8] Berentsen, Aleksander, Miguel Molico, and Randall Wright (2002). "Indivisibilities, Lotteries, and Monetary Exchange," *Journal of Economic Theory*, 107, 70-94.

[9] Berentsen, Aleksander, and Christopher Waller (2011). "Outside versus inside bonds: A Modigliani-Miller type result for liquidity constrained economies," *Journal of Economic Theory,* 146, 1852–1887.

[10] Brunnermeier, Markus (2009). "Deciphering the 2007-08 Liquidity and Credit Crunch," *Journal of Economic Perspectives*, 23, 77-100.

[11] Diamond, Douglas, and Philip Dybvig (1983). "Bank runs, deposit insurance and liquidity," *Journal of Political Economy*, 91, 401-419.

[12] Flannery, Mark (1996). "Financial Crises, Payments System Problems and Discount Window Lending," *Journal of Money, Credit and Banking*, 28, 804-824.

[13] Freixas, Xavier, and Jose Jorge (2008). "The Role of Interbank Markets in Monetary Policy: A Model with Rationing," *Journal of Money, Credit and Banking*, 40(6), 1151-1176.

[14] Gale, Douglas, and Tanju Yorulmazer (2011). "Liquidity Hoarding," Working Paper, Federal Reserve Bank of New York.

[15] Goodfriend, Marvin, and Robert King (1988). "Financial Deregulation, Monetary Policy and Central Banking," Federal Reserve Bank of Richmond Economic Review, 74(3).

[16] Guggenheim, Basil, Sébastian Kraenzlin, and Silvio Schumacher (2011). "Exploring an uncharted market: Evidence on the unsecured Swiss franc money market," Swiss National Bank Working Papers, 2011(5).

[17] Heider, Florian, Marie Hoerova, and Cornelia Holthausen (2008). "Liquidity Hoarding and Interbank Market Spreads: The Role of Counterparty Risk," Working Paper, European Central Bank.

[18] Ivashina, Victoria, and David Scharfstein (2010). "Bank lending during the financial crisis of 2008," *Journal of Financial Economics*, 97, 319-338.