
Functions and Taylor's formula

1. Let $f(x_1, x_2, x_3) = -x_1^2 + 6x_1x_2 - 9x_2^2 - 2x_3^2$. Determine the gradient and the Hesse matrix of f . Is f concave, convex, quasi-concave or quasi-convex? Compute the (general) 2nd Taylor polynomial of f . 5

2. Prove that if ϕ is twice continuously differentiable and $\phi(x, y) = c$ defines y as a twice differentiable function of x , then

$$y' = -\frac{\phi_x}{\phi_y} \quad \text{and} \quad y'' = -\frac{\phi_{xx} + 2\phi_{xy} \cdot y' + \phi_{yy} \cdot (y')^2}{\phi_y}.$$

Prove that

$$y'' = \frac{1}{(\phi_y)^3} \cdot \det \begin{pmatrix} 0 & \phi_x & \phi_y \\ \phi_x & \phi_{xx} & \phi_{xy} \\ \phi_y & \phi_{yx} & \phi_{yy} \end{pmatrix}.$$

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3. The following functions are important for economists ($x_1, x_2 \geq 0$):

• Cobb-Douglas functions: $f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$ with $\alpha_1, \alpha_2 > 0$; 5

• Quasilinear functions: $f(x_1, x_2) = x_1 + g(x_2)$
with $g : \mathbb{R} \rightarrow \mathbb{R}$ differentiable; 5

• Utility function: $f(x_1, x_2) = q_1 \cdot g(x_1) + q_2 \cdot g(x_2)$
with $g : \mathbb{R} \rightarrow \mathbb{R}$ differentiable and $q_1 + q_2 = 1$. 5

Compute $f_{x_1}, f_{x_2}, f_{x_i x_j}$ for all $1 \leq i, j \leq 2$, $\frac{f_{x_1}}{f_{x_2}}, \nabla f(\mathbf{a}), \nabla^2 f(\mathbf{a}), \det(\nabla^2 f(\mathbf{a}))$ and 2nd Taylor polynomial of f at $\mathbf{0}$ for all functions.

4. Prove that the function $f(x_1, x_2) = x_1 x_2$ is quasi-concave on $S = \mathbb{R}_{++}^2$.
Is f concave? 10