

Linear Algebra

1. The rank $rk(\mathbf{A})$ of a matrix \mathbf{A} is the maximal number of linearly independent column vectors in \mathbf{A} . Determine the rank of the following matrix for all values of the real parameter a . Hint: Use elementary row operations. 6

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 & 0 \\ -1 & a & -1 & a \\ 2 & 0 & 1 & -1 \end{pmatrix}$$

2. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

- (a) Determine $rk(\mathbf{A})$ and $\det(\mathbf{A})$. 2
- (b) Determine the eigenvalues and eigenvectors of \mathbf{A} . 4
- (c) Verify the **Spectral Theorem for symmetric matrices** for \mathbf{A} by finding a matrix \mathbf{P} : 4
- (d) Determine the Spectral Decomposition of \mathbf{A} . 4
- (e) Investigate the definiteness of \mathbf{A} . 2
3. Use the identity $\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$ for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ to prove the following facts:
- (a) $\det(\mathbf{A}^{-1}) = (\det(\mathbf{A}))^{-1}$; 3
- (b) $\det(\lambda \mathbf{A}) = \lambda^n \det(\mathbf{A})$ for $\lambda \in \mathbb{R}$; 3
- (c) \mathbf{A} and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ have the same eigenvalues. 6
4. Let \mathbf{A} be a square matrix with $\det \mathbf{A} \neq 0$ and let λ be an eigenvalue of \mathbf{A} . Prove:
- (a) $\lambda \neq 0$ and 2
- (b) $1/\lambda$ is an eigenvalue of the inverse \mathbf{A}^{-1} . 4