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 Static Optimization
 

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1. The following functions are important for economists ( $x_1, x_2 \geq 0$ ):

• Cobb-Douglas functions:  $f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$  with  $\alpha_1, \alpha_2 > 0$ ; 5

• Quasilinear functions:  $f(x_1, x_2) = x_1 + g(x_2)$  with  $g : \mathbb{R} \rightarrow \mathbb{R}$  differentiable; 5

• Utility function:  $f(x_1, x_2) = q_1 \cdot g(x_1) + q_2 \cdot g(x_2)$  with  $g : \mathbb{R} \rightarrow \mathbb{R}$  differentiable and  $q_1 + q_2 = 1$ . 5

Let  $D = \{(x_1, x_2) \in \mathbb{R}^2 \mid \phi(x_1, x_2) = p_1 x_1 + p_2 x_2 - I\}$  for  $p_1, p_2, I > 0$ . Compute all stationary points of the corresponding Lagrange functions  $l(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda \phi(x_1, x_2)$  for all functions. Use  $g(x) = x^k$ .

2. 5

$$\begin{aligned} \max \quad & : f(x, y, z) = \frac{1}{2}x - y \\ \text{subject to} \quad & : \begin{cases} g_1(x, y, z) = x - e^{-x} - y + z^2 \leq 0 \\ g_2(x, y, z) = -x \leq 0 \end{cases} \end{aligned}$$

3. 10

$$\begin{aligned} \max \quad & : f(x, y) = xy + x^2 \\ \text{subject to} \quad & : \begin{cases} g_1(x, y) = x^2 + y \leq 2 \\ g_2(x, y) = -y \leq 1 \end{cases} \end{aligned}$$

4. Let  $Q(\mathbf{x}) = Q(x_1, x_2) = \mathbf{x}^T \underbrace{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}}_{=\mathbf{A}} \mathbf{x} = ax_1^2 + bx_1x_2 + cx_2^2$

and  $D = \{(x_1, x_2) \in \mathbb{R}^2 \mid \phi(\mathbf{x}) = \mathbf{x}^T \mathbf{x} - r^2 = x_1^2 + x_2^2 - r^2 = 0\}$ .

Prove that  $Q$  attains maximum and minimum value over the set  $D$  which are related to the largest and smallest eigenvalues of  $\mathbf{A}$ . 10

The general case is:

Let  $\mathbf{A}$  be a symmetric  $n \times n$  matrix,  $Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$  a quadratic form and  $D = \{\mathbf{x} \in \mathbb{R}^n \mid \phi(\mathbf{x}) = \mathbf{x}^T \mathbf{x} - r^2 = 0\}$ . Prove that  $Q$  attains maximum and minimum value over the set  $D$  which are related to the largest and smallest eigenvalues of  $\mathbf{A}$ .