Fundamentals

Solve the following problems and hand in your solutions of the (∗) - marked questions. The solutions will be marked.

Matrices, eigenvalues and eigenvectors

1. Compute the (complex) eigenvalues and the associated (complex) eigenvectors for the following matrices:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

2. (∗) The rank $r(\mathbf{A})$ of a matrix $\mathbf{A}$ is the maximal number of linearly independent column vectors in $\mathbf{A}$. Determine the rank of the following matrix for all values of the real parameter a. Hint: Use elementary row operations.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 & 0 \\ -1 & a & -1 & a \\ 2 & 0 & 1 & -1 \end{pmatrix}$$

3. (∗) Verify the **Spectral Theorem for symmetric matrices** for the following matrix by finding a matrix $\mathbf{P}$:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

4. Use the identity $\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$ for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ to prove the following facts:

   (a) $\det(\mathbf{A}^{-1}) = (\det(\mathbf{A}))^{-1}$;

   (b) $\det(\lambda\, \mathbf{A}) = \lambda^n \det(\mathbf{A})$ for $\lambda \in \mathbb{R}$;

   (c) $\mathbf{A}$ and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ have the same eigenvalues.

5. (∗) Let $\mathbf{A}$ be a square matrix with $\det \mathbf{A} \neq 0$ and let λ be an eigenvalue of $\mathbf{A}$. Prove:

   (a) $\lambda \neq 0$ and

   (b) $1/\lambda$ is an eigenvalue of the inverse $\mathbf{A}^{-1}$.

6. (∗) Let $\mathbf{B}$ be an $n \times n$ matrix. Show that $\mathbf{A} = \mathbf{B}^T\mathbf{B}$ is positive semidfinite.

Functions and Taylor's formula

1. Let $f(x_1, x_2, x_3) = -x_1^2 + 6x_1x_2 - 9x_2^2 - 2x_3^2$. Determine the gradient and the Hesse matrix of f.

2. (*) Prove that if ϕ is twice countinuously differentiable and $\phi(x, y) = c$ defines y as a twice differentiable function of x, then

$$y' = -\frac{\phi_x}{\phi_y} \qquad \text{and} \qquad y'' = -\frac{\phi_{xx} + 2\phi_{xy} \cdot y' + \phi_{yy} \cdot (y')^2}{\phi_y}.$$

3. (*) The following functions are important for economists ($x_1, x_2 \geq 0$):

   - Cobb-Douglas functions: $f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$ with $\alpha_1, \alpha_2 > 0$;
   - Quasilinear functions: $f(x_1, x_2) = x_1 + g(x_2)$ with $g : \mathbb{R} \to \mathbb{R}$ differentiable;
   - Utility function: $f(x_1, x_2) = q_1 \cdot g(x_1) + q_2 \cdot g(x_2)$ with $g : \mathbb{R} \to \mathbb{R}$ differentiable and $q_1 + q_2 = 1$.

   Questions:

   (a) Compute $\dfrac{f_{x_1}}{f_{x_2}}$ for all functions.

   (b) Compute $\nabla f(\mathbf{a})$ and $\nabla^2 f(\mathbf{a})$, $\det(\nabla^2 f(\mathbf{a}))$ and the sign of $\det(\nabla^2 f(\mathbf{a}))$ for all functions.

   (c) Compute the 2-nd Taylor polynomial $P_2(\mathbf{x}, \mathbf{a})$ of f in $\mathbf{a}$ for all functions.

4. (*) Let $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ be homogeneous of degree d. Prove the following fact: At each point on a given ray through the origin the gradients of f are proportional.