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**Functions and Taylor's formula**


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- Let  $f(x_1, x_2, x_3) = -x_1^2 + 6x_1x_2 - 9x_2^2 - 2x_3^2$ . Determine the gradient and the Hesse matrix of  $f$ . Is  $f$  concave, convex, quasi-concave or quasi-convex? Compute the (general) 2nd Taylor polynomial of  $f$ .
- Prove that if  $\phi$  is twice continuously differentiable and  $\phi(x, y) = c$  defines  $y$  as a twice differentiable function of  $x$ , then

$$y' = -\frac{\phi_x}{\phi_y} \quad \text{and} \quad y'' = -\frac{\phi_{xx} + 2\phi_{xy} \cdot y' + \phi_{yy} \cdot (y')^2}{\phi_y}.$$

Prove that

$$y'' = \frac{1}{(\phi_y)^3} \cdot \det \begin{pmatrix} 0 & \phi_x & \phi_y \\ \phi_x & \phi_{xx} & \phi_{xy} \\ \phi_y & \phi_{yx} & \phi_{yy} \end{pmatrix}.$$

- The following functions are important for economists ( $x_1, x_2 \geq 0$ ):
  - Cobb-Douglas functions:  $f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$  with  $\alpha_1, \alpha_2 > 0$ ;
  - Quasilinear functions:  $f(x_1, x_2) = x_1 + g(x_2)$  with  $g : \mathbb{R} \rightarrow \mathbb{R}$  differentiable;
  - Utility function:  $f(x_1, x_2) = q_1 \cdot g(x_1) + q_2 \cdot g(x_2)$  with  $g : \mathbb{R} \rightarrow \mathbb{R}$  differentiable and  $q_1 + q_2 = 1$ .

Questions:

- Compute  $\frac{f_{x_1}}{f_{x_2}}$  for all functions.
  - Compute  $\nabla f(\mathbf{a})$  and  $\nabla^2 f(\mathbf{a})$ ,  $\det(\nabla^2 f(\mathbf{a}))$  and the sign of  $\det(\nabla^2 f(\mathbf{a}))$  for all functions.
  - Compute the 2-nd Taylor polynomial  $P_2(\mathbf{x}, \mathbf{a})$  of  $f$  in  $\mathbf{a}$  for all functions.
- Let  $f(\mathbf{x}) = f(x_1, \dots, x_n)$  be homogeneous of degree  $d$ . Prove the following fact: At each point on a given ray through the origin the gradients of  $f$  are proportional.
  - Prove that the function  $f(x_1, x_2) = x_1x_2$  is quasi-concave on  $S = \mathbb{R}_{++}^2$ .
  - Let  $\lambda_1, \lambda_2 \in \mathbb{R}$ . Verify that the function  $f(x_1, x_2) = \lambda_1x_1 + \lambda_2x_2$  is concave, convex, quasi-concave and quasiconvex. Is it strictly any of these? Sketch its level sets.