Dr. Thomas Zehrt **Advanced Mathematics for Economics** Exercise **2**

Functions and Taylor's formula

1. Let $f(x_1, x_2, x_3) = -x_1^2 + 6x_1x_2 - 9x_2^2 - 2x_3^2$. Determine the gradient and the Hesse matrix of f. Is f concave, convex, quasi-concave or quasi-convex? Compute the (gerneral) 2nd Taylor polynomial of f.

2. Prove that if ϕ is twice countinuously differentiable and $\phi(x, y) = c$ defines y as a twice differentiable function of x, then

$$y' = -\frac{\phi_x}{\phi_y} \qquad \text{and} \qquad y'' = -\frac{\phi_{xx} + 2\phi_{xy} \cdot y' + \phi_{yy} \cdot (y')^2}{\phi_y}.$$

   Prove that

$$y'' = \frac{1}{(\phi_y)^3} \cdot \det \begin{pmatrix} 0 & \phi_x & \phi_y \\ \phi_x & \phi_{xx} & \phi_{xy} \\ \phi_y & \phi_{yx} & \phi_{yy} \end{pmatrix}.$$

3. The following functions are important for economists ($x_1, x_2 \geq 0$):
   - Cobb-Douglas functions: $f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$ with $\alpha_1, \alpha_2 > 0$;
   - Quasilinear functions: $f(x_1, x_2) = x_1 + g(x_2)$ with $g : \mathbb{R} \to \mathbb{R}$ differentiable;
   - Utility function: $f(x_1, x_2) = q_1 \cdot g(x_1) + q_2 \cdot g(x_2)$ with $g : \mathbb{R} \to \mathbb{R}$ differentiable and $q_1 + q_2 = 1$.

   Compute $\frac{f_{x_1}}{f_{x_2}}$, $\nabla f(\mathbf{a})$, $\nabla^2 f(\mathbf{a})$, $\det(\nabla^2 f(\mathbf{a}))$ and the sign of $\det(\nabla^2 f(\mathbf{a}))$ for all functions.

4. Let $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ be homogeneous of degree d. Prove the following fact: At each point on a given ray through the origin the gradients of f are proportional.

5. Prove that the function $f(x_1, x_2) = x_1x_2$ is quasi-concave on $S = \mathbb{R}^2_{++}$.

6. Let $\lambda_1, \lambda_2 \in \mathbb{R}$. Verify that the function $f(x_1, x_2) = \lambda_1 x_1 + \lambda_2 x_2$ is concave, convex, quasi-concave and quasiconvex. Is it strictly any of these? Sketch its level sets.