Dr. Thomas Zehrt **Advanced Mathematics for Economics** Exercise **1**

**Linear Algebra**

1. Compute the (complex) eigenvalues and the associated (complex) eigenvectors for the following matrices:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

2. Complete the square for the quadratic form

$$Q(x_1, x_2) = (x_1\ x_2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Hence find conditions on $a, b, c$ which are necessary and sufficient for postive definitness/negative definitness.

3. The rank $r(\mathbf{A})$ of a matrix $\mathbf{A}$ is the maximal number of linearly independent column vectors in $\mathbf{A}$. Determine the rank of the following matrix for all values of the real parameter $a$. Hint: Use elementary row operations.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 & 0 \\ -1 & a & -1 & a \\ 2 & 0 & 1 & -1 \end{pmatrix}$$

4. Verify the **Spectral Theorem for symmetric matrices** for the following matrix by finding a matrix $\mathbf{P}$:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Investigate the definitmess of $\mathbf{A}$.

5. Use the identity $\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$ for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ to prove the following facts:

   (a) $\det(\mathbf{A}^{-1}) = (\det(\mathbf{A}))^{-1}$;

   (b) $\det(\lambda\, \mathbf{A}) = \lambda^n\ \det(\mathbf{A})$ for $\lambda \in \mathbb{R}$;

   (c) $\mathbf{A}$ and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ have the same eigenvalues.

6. Let $\mathbf{A}$ be a square matrix with $\det \mathbf{A} \neq 0$ and let $\lambda$ be an eigenvalue of $\mathbf{A}$. Prove:

   (a) $\lambda \neq 0$ and

   (b) $1/\lambda$ is an eigenvalue of the inverse $\mathbf{A}^{-1}$.

7. Let $\mathbf{B}$ be an $n \times n$ matrix. Show that $\mathbf{A} = \mathbf{B}^T\mathbf{B}$ is positive semidfinite. Let $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{B}\mathbf{x}$ where $\mathbf{B}$ is not symmetric. Let $\mathbf{A} = (\mathbf{B} + \mathbf{B}^T)/2$ and $\mathbf{C} = (\mathbf{B} - \mathbf{B}^T)/2$. Show that $\mathbf{A}$ is symmetric and evaluate both $\mathbf{x}^T\mathbf{A}\mathbf{x}$ and $\mathbf{x}^T\mathbf{C}\mathbf{x}$.