Static Optimization

1. The following functions are important for economists ($x_1, x_2 \geq 0$):
   - Cobb-Douglas functions: $f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$ with $\alpha_1, \alpha_2 > 0$;
   - Quasilinear functions: $f(x_1, x_2) = x_1 + g(x_2)$ with $g : \mathbb{R} \to \mathbb{R}$ differentiable;
   - Utility function: $f(x_1, x_2) = q_1 \cdot g(x_1) + q_2 \cdot g(x_2)$ with $g : \mathbb{R} \to \mathbb{R}$ differentiable and $q_1 + q_2 = 1$.

   Let $D = \{(x_1, x_2) \in \mathbb{R}^2 \mid \phi(x_1, x_2) = p_1 x_1 + p_2 x_2 - I\}$ for $p_1, p_2, I > 0$. Compute all stationary points of the corresponding Lagrange functions $l(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda\phi(x_1, x_2)$ for all functions. Use $g(x) = x^k$.

2. Let $Q(\mathbf{x}) = Q(x_1, x_2) = \mathbf{x}^T \underbrace{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}}_{=\mathbf{A}} \mathbf{x} = a x_1^2 + b x_1 x_2 + c x_2^2$

   and $D = \{(x_1, x_2) \in \mathbb{R}^2 \mid \phi(\mathbf{x}) = \mathbf{x}^T\mathbf{x} - r^2 = x_1^2 + x_2^2 - r^2 = 0\}$.

   Prove that Q attains maximum and minimum value over the set D which are related to the largest and smallest eigenvalues of $\mathbf{A}$.

   Or prove the general case:

   Let $\mathbf{A}$ be a symmetric $n \times n$ matrix, $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ a quadratic form and $D = \{\mathbf{x} \in \mathbb{R}^n \mid \phi(\mathbf{x}) = \mathbf{x}^T\mathbf{x} - r^2 = 0\}$. Prove that Q attains maximum and minimum value over the set D which are related to the largest and smallest eigenvalues of $\mathbf{A}$.

3. Suppose that $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $n \times n$ symmetric positive definite matrix. Show directly that

   $$Q(\mathbf{y} + t(\mathbf{x} - \mathbf{y})) - tQ(\mathbf{x}) - (1-t)Q(\mathbf{y}) \leq 0$$

   for all $t \in [0, 1]$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

4.
   $$\max \quad : f(x, y, z) = \frac{1}{2}x - y$$

   $$\text{subject to} \quad : \begin{cases} g_1(x, y, z) = x - e^{-x} - y + z^2 \leq 0 \\ g_2(x, y, z) = -x \leq 0 \end{cases}$$

5.
   $$\max \quad : f(x, y) = xy + x^2$$

   $$\text{subject to} \quad : \begin{cases} g_1(x, y) = x^2 + y \leq 2 \\ g_2(x, y) = -y \leq 1 \end{cases}$$