Dr. Thomas Zehrt **Advanced Mathematics for Economics** Exercise **1**

Linear Algebra

1. The rank $r(\mathbf{A})$ of a matrix $\mathbf{A}$ is the maximal number of linearly independent column vectors in $\mathbf{A}$. Determine the rank of the following matrix for all values of the real parameter a. Hint: Use elementary row operations.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 & 0 \\ -1 & a & -1 & a \\ 2 & 0 & 1 & -1 \end{pmatrix}$$

2. Verify the **Spectral Theorem for symmetric matrices** for the following matrix by finding a matrix $\mathbf{P}$:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Investigate the definitmess of $\mathbf{A}$.

3. Use the identity $\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$ for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ to prove the following facts:

   (a) $\det(\mathbf{A}^{-1}) = (\det(\mathbf{A}))^{-1}$;

   (b) $\det(\lambda\, \mathbf{A}) = \lambda^n \det(\mathbf{A})$ for $\lambda \in \mathbb{R}$;

   (c) $\mathbf{A}$ and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ have the same eigenvalues.

4. Let $\mathbf{A}$ be a square matrix with $\det \mathbf{A} \neq 0$ and let λ be an eigenvalue of $\mathbf{A}$. Prove:

   (a) $\lambda \neq 0$ and

   (b) $1/\lambda$ is an eigenvalue of the inverse $\mathbf{A}^{-1}$.