Linear algebra

Keywords: vector, matrix, eigenvalue, eigenvector, diagonalization, linear transformation, quadratic forms and symmetric matrices

Compare: Vorlesung Mathematik 2

Contents

1 Matrices and vectors

1.1 Real Vectors

- n-dimensional space $\mathbb{R}^n$
- elements $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are called <u>n-vectors</u>

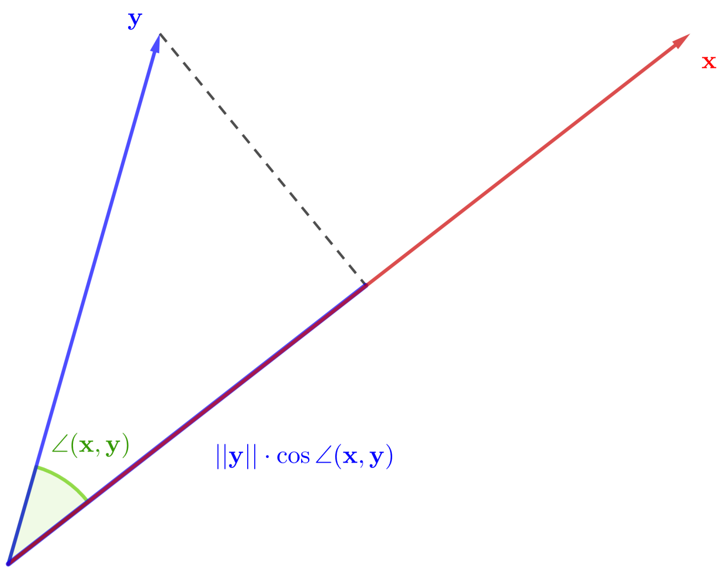
$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}^T \quad \text{and } \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

- <u>scalar product</u> and <u>norm</u>:

$$\begin{aligned} \mathbf{x} \bullet \mathbf{y} \ = \ \mathbf{x}^T\mathbf{y} \ = < \mathbf{x}, \mathbf{y} > &= x_1y_1 + x_2y_2 + \dots + x_ny_n \\ ||\mathbf{x}|| \ = \ \sqrt{\mathbf{x} \bullet \mathbf{x}} &= \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \end{aligned}$$

$$\boxed{\mathbf{x} \bullet \mathbf{y} \ = \ ||\mathbf{x}|| \cdot ||\mathbf{y}|| \cdot \cos \angle(\mathbf{x}, \mathbf{y})}$$

You may see, that $||\mathbf{y}|| \cdot \cos \angle(\mathbf{x}, \mathbf{y})$ is the length of the orthogonal projection of the vector $\mathbf{y}$ on $\mathbf{x}$, with the negative sign if the projection has an opposite direction with respect to $\mathbf{x}$.



- Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathbb{R}^n$ be a family of vectors.
  - If $a_1, a_2, \ldots, a_k \in \mathbb{R}$, then $\mathbf{z} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k$ is called a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$.
  - The set of all linear combinations of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is called the vector space spanned by the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ and denoted by

    $$V(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) \;=\; \{a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k \mid a_1, a_2, \ldots, a_k \in \mathbb{R}\}$$

  - $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are called linearly dependent, if there exist $b_1, b_2, \ldots, b_k \in \mathbb{R}$ such that $b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_k\mathbf{x}_k = \mathbf{0}$ and not all $b_j = 0$.
  - $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are called linearly independent, if a linear combination of the zero vector

    $$b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_k\mathbf{x}_k = \mathbf{0}$$

    is possible only with $b_1 = b_2 = \cdots = b_k = 0$.
  - Each family of exactly n linearly independent vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^n$ is a so called basis of $\mathbb{R}^n$. This means, that each vector $\mathbf{x} \in \mathbb{R}^n$ can uniquely expressed as a linear combination of the basis:

    $$\mathbf{x} \;=\; b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_n\mathbf{x}_n$$

  - A family of n (linearly independent) vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n \in \mathbb{R}^n$ is called orthonormal basis of $\mathbb{R}^n$ if

    $$\mathbf{p}_i \bullet \mathbf{p}_j \;=\; \mathbf{p}_i^T\mathbf{p}_j \;=\; \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

    for all $i, j = 12, \ldots, n$. This means, that each vector has length 1 and each pair of (different) vectors has a right angle. As before, each vector $\mathbf{x} \in \mathbb{R}^n$ can uniquely expressed as a linear combination of the orthonormal basis

    $$\mathbf{x} \;=\; b_1\mathbf{p}_1 + b_2\mathbf{p}_2 + \cdots + b_n\mathbf{p}_n \;=\; \sum_{i=1}^{n} b_i\mathbf{p}_i$$

    but the coefficients b_i have a nice interpretation (for orthonormal bases). We see

    $$\mathbf{p}_j^T\mathbf{x} \;=\; \sum_{i=1}^{n} b_i\mathbf{p}_j^T\mathbf{p}_i \;=\; b_j\mathbf{p}_j^T\mathbf{p}_j \;=\; b_j$$

    Hence the coefficient

    $$b_j = \mathbf{p}_j^T\mathbf{x} = ||\mathbf{p}_j|| \cdot ||\mathbf{x}|| \cdot \cos\angle(\mathbf{p}_j, \mathbf{x}) = ||\mathbf{x}|| \cdot \cos\angle(\mathbf{p}_j, \mathbf{x})$$

    is the length of the orthogonal projection of the vector $\mathbf{x}$ on the basis vector $\mathbf{p}_j$.

1.2 Real Matrices

$\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n$

$$\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}, \ldots, \mathbf{a}_m = \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix}$$

$$\rightarrow \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{pmatrix} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m)$$

is called an $n \times m$ matrix.
Notation: $\mathbf{A} \in \mathbb{R}^{n \times m}$

- The inverse matrix $\mathbf{A}^{-1}$ of the $n \times n$ matrix $\mathbf{A} = (a_{ij})$ is defined by

$$\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{I_n} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

- For the $n \times n$ matrix $\mathbf{A}$ let $\mathbf{A}_{ij}$ denote the $(n-1) \times (n-1)$ submatrix of $\mathbf{A}$ generated by cancelling the i-th row and the j-th column of $\mathbf{A}$. Then the determinant $\det(\mathbf{A})$ is given (recursively) by the so called cofactor expansion

$$\boxed{\det(\mathbf{A}) = |\mathbf{A}| = a_{11} \det \mathbf{A}_{11} - a_{12} \det \mathbf{A}_{12} + \cdots + (-1)^{n+1} a_{1n} \det \mathbf{A}_{1n}}$$

- $\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$

Example 1.1

$$\begin{vmatrix} 1 & 1 & 3 & 3 \\ 1 & 2 & 1 & 2 \\ 1 & -2 & 1 & -2 \\ 0 & 1 & -2 & -1 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 2 & 1 & 2 \\ -2 & 1 & -2 \\ 1 & -2 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 0 & -2 & -1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 2 & 2 \\ 1 & -2 & -2 \\ 0 & 1 & -1 \end{vmatrix} - 3 \cdot \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix}.$$

1.3 Linear transformations and matrices

Definition 1.1 *A linear transformation is a map $T : \mathbb{R}^m \to \mathbb{R}^n$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ and all $\lambda, \mu \in \mathbb{R}$ we have:*

$$\boxed{T(\lambda \cdot \mathbf{x} + \mu \cdot \mathbf{y}) \;=\; \lambda \cdot T(\mathbf{x}) + \mu \cdot T(\mathbf{y})}$$

Example 1.2

- *The map $T(\mathbf{x}) = T(x_1, x_2, x_3) = x_1 + 2x_2 + 4x_3$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^1$:*

$$\begin{aligned} T(\lambda \cdot \mathbf{x} + \mu \cdot \mathbf{y}) &= T(\lambda \cdot x_1 + \mu \cdot y_1, \lambda \cdot x_2 + \mu \cdot y_2, \lambda \cdot x_3 + \mu \cdot y_3) \\ &= \lambda \cdot x_1 + \mu \cdot y_1 + 2 \cdot (\lambda \cdot x_2 + \mu \cdot y_2) + 4 \cdot (\lambda \cdot x_3 + \mu \cdot y_3) \\ &= \lambda \cdot x_1 + 2 \cdot \lambda \cdot x_2 + 4 \cdot \lambda \cdot x_3 + \mu \cdot y_1 + 2 \cdot \mu \cdot y_2 + 4 \cdot \mu \cdot y_3 \\ &= \lambda \cdot (x_1 + 2 \cdot x_2 + 4 \cdot x_3) + \mu \cdot (y_1 + 2 \cdot y_2 + 4 \cdot y_3) \\ &= \lambda \cdot T(\mathbf{x}) + \mu \cdot L(\mathbf{y}) \end{aligned}$$

- *The map $L(\mathbf{x}) = L(x_1, x_2) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Proof it!*

- *The map $L(\mathbf{x}) = L(x_1, x_2) = \begin{pmatrix} x_1^2 + x_2 \\ x_1 - x_2 \end{pmatrix}$ is* **not** *linear.*

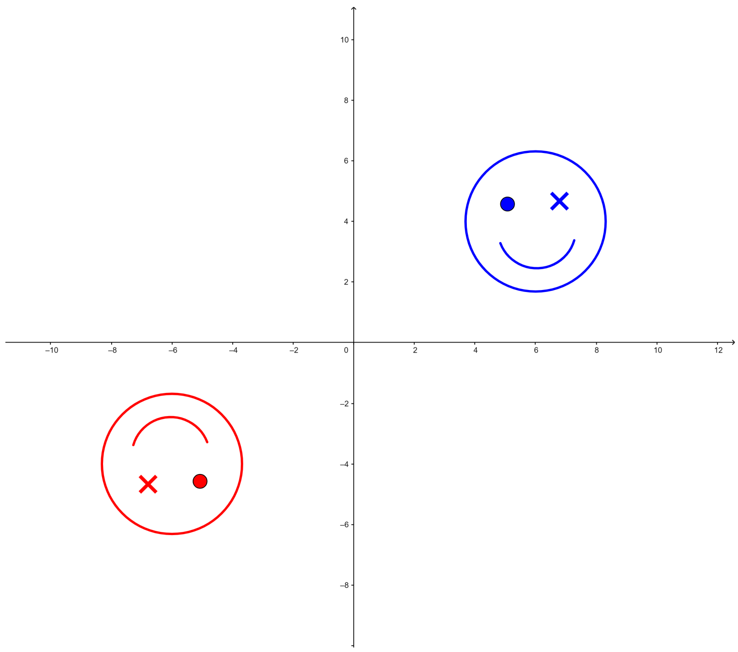
Each $n \times m$ matrix $\mathbf{A}$ defines a linear transformation by matrix multiplication

$$\begin{aligned} T_{\mathbf{A}}(\mathbf{x}) \;=\; \mathbf{A} \cdot \mathbf{x} &= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{pmatrix} \\ &= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + x_m \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix} \end{aligned}$$

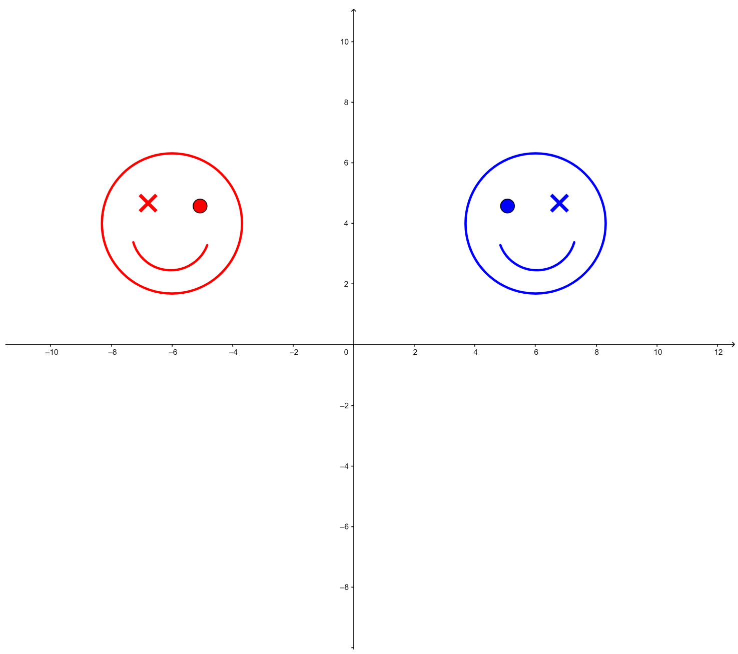
The image of the vector $\mathbf{x} \in \mathbb{R}^m$ is a linear combination of the column vectors of the matrix $\mathbf{A}$.

Example 1.3 *In the following picture you can see the original figur (blue) and the image of this figur under the linear map L_A. Each blue point (endpoint of the vector $\mathbf{x}$) is maped on the point $A\mathbf{x}$ (red).*

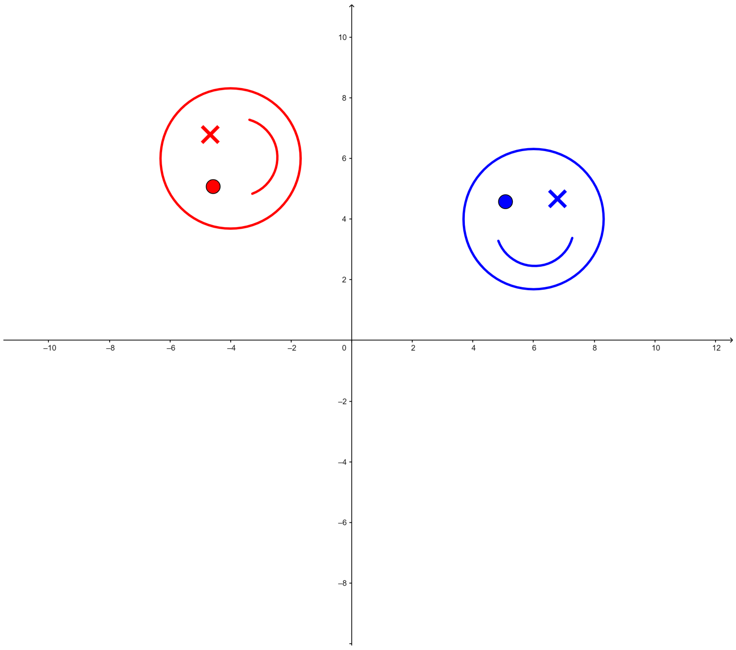
- $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ *Rotation with center* $(0,0)$ *by* 180 *degree*



- $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ *Reflection along the* y*-axis*



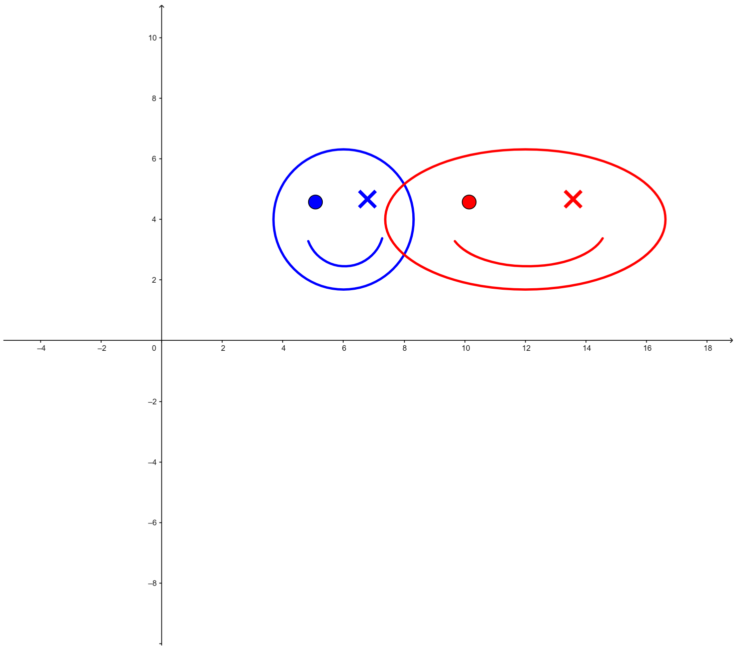
- $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *Rotation with center* $(0,0)$ *by* 90 *degree*



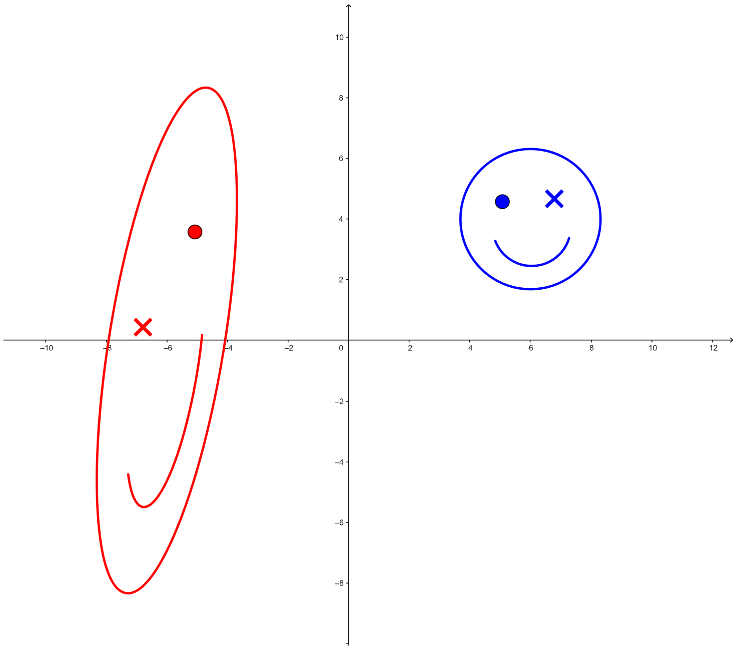
Remark: *The general rotation with center* $(0,0)$ *by* α *degree is given by the following matrix:*

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

- $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ *Scaling (by the factor* 2*) in* x*-direction*



- $\begin{pmatrix} -1 & 0 \\ -2 & 3 \end{pmatrix}$?



Projections on lines The following type of matrices is of the special interest. Let

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$$

be an arbitrary non-zero vector. The direct calculation

$$\begin{aligned} \mathbf{p} \cdot \mathbf{p}^T &= \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \cdot \begin{pmatrix} p_1 & p_2 & \dots & p_n \end{pmatrix} = \begin{pmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & \dots & p_1 \cdot p_n \\ p_2 \cdot p_1 & p_2 \cdot p_2 & \dots & p_2 \cdot p_n \\ \vdots & \vdots & \vdots & \vdots \\ p_n \cdot p_1 & p_n \cdot p_2 & \dots & p_n \cdot p_n \end{pmatrix} \\ &= \begin{pmatrix} - & p_1 \cdot \mathbf{p}^T & - \\ - & p_2 \cdot \mathbf{p}^T & - \\ \vdots & \vdots & \vdots \\ - & p_2 \cdot \mathbf{p}^T & - \end{pmatrix} = \begin{pmatrix} | & | & & | \\ p_1 \cdot \mathbf{p} & p_2 \cdot \mathbf{p} & \dots & p_n \cdot \mathbf{p} \\ | & | & & | \end{pmatrix} \end{aligned}$$

shows that $\mathbf{p} \cdot \mathbf{p}^T$ is a symmetric $n \times n$ matrix of rank 1 (all columns and all rows are multiples of the vector $\mathbf{p}$ resp. $\mathbf{p}^T$).

Theorem 1.1 *Let $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n \in \mathbb{R}^n$ an orthonormal basis of $\mathbb{R}^n$, this means*

$$\mathbf{p}_i \bullet \mathbf{p}_j = \mathbf{p}_i^T \mathbf{p}_j = \begin{cases} 0 & \textit{if } i \neq j \\ 1 & \textit{if } i = j \end{cases}$$

and $\mathbf{P}_i = \mathbf{p}_i \cdot \mathbf{p}_i^T$ for all $i, j = 1, 2, \dots, n$.
Then the linear map

$$T_{\mathbf{P}_i}(\mathbf{x}) = \mathbf{P}_i \cdot \mathbf{x}$$

given by matrix multiplication is a projection on the line spanned by the vector $\mathbf{p}_i$ for all $i = 1, 2, \dots, n$.

Proof: Let

$$\mathbf{x} = b_1 \mathbf{p}_1 + b_2 \mathbf{p}_2 + \dots + b_n \mathbf{p}_n = \sum_{j=1}^{n} b_j \mathbf{p}_j$$

be a vector expressed in the given orthonormal basis. Then by direct calculation

$$\begin{aligned} \mathbf{P}_i \cdot \mathbf{x} &= \left(\mathbf{p}_i \cdot \mathbf{p}_i^T\right) \cdot \left(\sum_{j=1}^{n} b_j \mathbf{p}_j\right) \\ &= \mathbf{p}_i \cdot \left(\mathbf{p}_i^T \cdot \sum_{j=1}^{n} b_j \mathbf{p}_j\right) \\ &= \mathbf{p}_i \cdot \left(\sum_{j=1}^{n} b_j \mathbf{p}_i^T \cdot \mathbf{p}_j\right) = \mathbf{p}_i b_i = b_i \mathbf{p}_i. \end{aligned}$$

□

1.4 Complex matrices and vectors

Sometimes it is helpful to allow complex matrices and vectors (matrices whose elements are complex numbers). A complex matrix can be viewed as a combination of two real matrices:

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} a_{11}+ib_{11} & a_{12}+ib_{12} & \dots & a_{1m}+ib_{1m} \\ a_{21}+ib_{21} & a_{22}+ib_{22} & \dots & a_{2m}+ib_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}+ib_{n1} & a_{n2}+ib_{n2} & \dots & a_{nm}+ib_{nm} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} + i \cdot \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{pmatrix} \end{aligned}$$

1.5 Matrix calculus

1a. $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$

1b. In general: $\mathbf{AB} \neq \mathbf{BA}$

2a. $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$

2b. $(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$

3a. $\mathbf{A}+\mathbf{0}=\mathbf{A}$

3b. $\mathbf{A}I=I\mathbf{A}=\mathbf{A}$ ($\mathbf{A}$ square)

4. $\mathbf{AB}=\mathbf{0} \quad \not\Rightarrow \quad \mathbf{A}=\mathbf{0}$ or $\mathbf{B}=\mathbf{0}$

5. $\mathbf{AB}=\mathbf{AC} \quad \not\Rightarrow \quad \mathbf{B}=\mathbf{C}$

6. $\lambda(\mathbf{A}+\mathbf{B}) = \lambda\mathbf{A}+\lambda\mathbf{B} \quad \lambda \in \mathbb{R}$

7. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$

8. $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$

9. $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$

10. $(\mathbf{AB})^{\mathbf{-1}} = \mathbf{B}^{\mathbf{-1}}\mathbf{A}^{\mathbf{-1}}$

11. $(\mathbf{A}^T)^T = \mathbf{A}$

12. $(\mathbf{A}+\mathbf{B})^{\mathbf{T}} = \mathbf{A}^T+\mathbf{B}^{\mathbf{T}}$

13. $(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$

14. $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$

For $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad-bc \neq 0$ is $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

All these definitions and results can be generalized to vectors and matrices with complex entries.

2 Eigenvalues and eigenvectors

2.1 Definition and determination

Definition 2.1 *If* $\mathbf{A}$ *is a real (or complex)* $n \times n$ *matrix, then a (complex) number* λ *is an eigenvalue of* $\mathbf{A}$ *if there is a nonzero (complex) vector* $\mathbf{x} \in \mathbb{C}^{\mathbf{n}}$ *such that*

$$\boxed{\mathbf{A}\mathbf{x} \;=\; \lambda\mathbf{x}}$$

Then $\mathbf{x}$ *is an eigenvector of* $\mathbf{A}$ *(associated with* λ*).*

Remark: If $\mathbf{x}$ is an eigenvector associated with the eigenvalue λ, then so is $\alpha\mathbf{x}$ for every real (and complex) number $\alpha \neq 0$.

$$\mathbf{A}\,(\alpha\,\mathbf{x}) \;=\; \alpha\,\mathbf{A}\,\mathbf{x} \;=\; \alpha\,(\lambda\,\mathbf{x}) \;=\; \lambda\,(\alpha\,\mathbf{x})$$

How to find eigenvalues? The equation can be written as

$$\begin{aligned} \mathbf{A}\,\mathbf{x} &= \lambda\,\mathbf{x} \\ \Leftrightarrow\quad \mathbf{A}\,\mathbf{x} - \lambda\,\mathbf{I}\,\mathbf{x} &= \mathbf{0} \\ \Leftrightarrow\quad (\mathbf{A} - \lambda\,\mathbf{I})\,\mathbf{x} &= \mathbf{0} \end{aligned}$$

This is a homogeneous system of linear equations. It has a solution $\mathbf{x} \neq \mathbf{0}$ if and only if the matrix $(\mathbf{A} - \lambda\,\mathbf{I})$ is singular which means that its determinant equals to 0.

$$\boxed{(\mathbf{A} - \lambda\,\mathbf{I}) \text{ singular} \quad\Leftrightarrow\quad \underbrace{\det(\mathbf{A} - \lambda\,\mathbf{I})}_{p_A(\lambda)} = 0}$$

$p_A(\lambda) = 0$ is called the characteristic equation of $\mathbf{A}$. The function $p_A(\lambda)$ is a polynomial of degree n in λ, called the characteristic polynomial of $\mathbf{A}$.

Theorem 2.1 *Are both* $\mathbf{x}$ *and* $\mathbf{y}$ *eigenvectors of* $\mathbf{A}$ *associated with the same eigenvalue* λ*, then all linear combinations of* $\mathbf{x}$ *and* $\mathbf{y}$ *are eigenvectors associated with* λ *to. This means, that the set of all eigenvectors (and the* $\mathbf{0}$*-vector) associated with an eigenvalue* λ *is a vector space, called the eigenspace of* λ*:*

$$V(\lambda) \;=\; \{\, \mathbf{x} \in \mathbb{C}^n \mid (\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0} \,\}.$$

The dimension of the vector space $V(\lambda)$ *is called the geometric multiplicity of the eigenvalue* λ*.*

Proof: Let $\mathbf{A}\mathbf{x} = \lambda\,\mathbf{x}$ and $\mathbf{A}\mathbf{y} = \lambda\,\mathbf{y}$ and $a, b \in \mathbb{R}$, not both equal to 0. Then we have for $\mathbf{z} = a\mathbf{x} + b\mathbf{y}$:

$$\mathbf{A}\mathbf{z} \;=\; \mathbf{A}(a\mathbf{x} + b\mathbf{y}) \;=\; a\mathbf{A}\mathbf{x} + b\mathbf{A}\mathbf{y} \;=\; a\lambda\mathbf{x} + b\lambda\mathbf{y} \;=\; \lambda\mathbf{z}.$$

□

Determination of the eigenvalues and eigenvectors

1. The polynomial equation $p_A(\lambda) = 0$ has always n complex solutions (counted with multiplicity) and may have no real solutions. If $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ are the pairwise distinct solutions (the eigenvalues of $\mathbf{A}$) with the multiplicities $k_1, \dots, k_r$ then the characteristic polynomial can be written as

$$p_A(\lambda) \quad = \quad (\lambda_1 - \lambda)^{k_1} \, (\lambda_2 - \lambda)^{k_2} \dots (\lambda_r - \lambda)^{k_r}.$$

   The multiplicity k_i of the zero λ_i is called algebraic multiplicity of the eigenvalue λ_i. Generally, the determination of the (exact) zeros is impossible for $n \geq 5$ and we have to use numerical methods.

2. For each eigenvalue λ_i $(1 \leq i \leq r)$ we compute the eigenspace of λ_i

$$V(\lambda_i) \quad = \quad \{ \, \mathbf{x} \in \mathbb{C}^n \mid (\mathbf{A} - \lambda_i \mathbf{I}) \, \mathbf{x} = \mathbf{0} \, \}.$$

Example 2.1

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- $p_A(\lambda) \; = \; (2 - \lambda)^2 (1 - \lambda)$
- *Zeros of the characteristic polynomial:* $\lambda_1 = 1$ *(algebraic multiplicity* 1*),* $\lambda_2 = 2$ *(algebraic multiplicity* 2*)*
- $$\left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \mathbf{x^{(1)}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

  and $V(-1) = \{ \, t \cdot \mathbf{x^{(1)}} \mid t \in \mathbb{R} \, \}$ *with geometric multiplicity* 1*.*
- $$\left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

  The 2*-dimensional vectorspace of all solutions is given by the single equation* $x_3 = 0$ *and there are infinitely many pairs of orthogonal vectors which span this space. We take the two standard vectors:*

$$V(2) \quad = \quad \left\{ t_1 \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{x^{(2)}}} + t_2 \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\mathbf{x^{(3)}}} \mid t_1, t_2 \in \mathbb{R} \right\}$$

Example 2.2

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{pmatrix}$$

- $p_A(\lambda) \;=\; -\lambda^3 + 4\lambda^2 - \lambda - 6 \;=\; (\lambda+1)\cdot(-\lambda^2 + 5\lambda - 6) \;=\; -(\lambda+1)\cdot(\lambda-2)\cdot(\lambda-3)$
- *Zeros of the characteristic polynomial:* $\lambda_1 = -1$, $\lambda_2 = 2$ *and* $\lambda_3 = 3$ *(all of algebraic multiplicity* 1*)*
- $$\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \;\to\; \mathbf{x^{(1)}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

  and $V(-1) = \{\, t \cdot \mathbf{x^{(1)}} \mid t \in \mathbb{R} \,\}$ *with geometric multiplicity* 1.
- $$\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \;\to\; \mathbf{x^{(2)}} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

  and $V(2) = \{\, t \cdot \mathbf{x^{(2)}} \mid t \in \mathbb{R} \,\}$ *with geometric multiplicity* 1.
- $$\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \;\to\; \mathbf{x^{(3)}} = \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

  and $V(3) = \{\, t \cdot \mathbf{x^{(3)}} \mid t \in \mathbb{R} \,\}$ *with geometric multiplicity* 1.

Definition 2.2 *The spectral radius of a quadratic matrix A is the real number*

$$\rho(A) \;:=\; \max\{|\lambda_1|, \ldots, |\lambda_r|\,\}.$$

2.2 *Generalized Eigenvectors*

To solve some interesting problems we have to generalize the notion of eigenvectors.

Definition 2.3 *A vector* $\mathbf{x} \in \mathbb{C}^n$ *is called* <u>*generalized eigenvector*</u> *of degree* $l \in \mathbb{N}$ *associated to the eigenvalue* λ *of* $\mathbf{A}$*, if*

$$(\mathbf{A} - \lambda\mathbf{I})^l \, \mathbf{x} \;=\; \mathbf{0} \quad \textit{and} \quad (\mathbf{A} - \lambda\mathbf{I})^{l-1} \, \mathbf{x} \;\neq\; \mathbf{0}.$$

Of course, an eigenvector is a generalized eigenvector of degree 1.

Example 2.3 *The matrix*

$$\mathbf{A} \;=\; \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

has the eigenvalue 1 *of (algebraic) multiplicity* 3 *with* $\dim V(1) = 1$ *(geometric multiplicity). We have:*

$$\begin{array}{lll} (\mathbf{A} - \mathbf{I})\, \mathbf{e}_1 \;=\; \mathbf{0} & (\mathbf{A} - \mathbf{I})\, \mathbf{e}_2 \;=\; \mathbf{e}_1 & (\mathbf{A} - \mathbf{I})^2\, \mathbf{e}_2 \;=\; \mathbf{0} \\ (\mathbf{A} - \mathbf{I})\, \mathbf{e}_3 \;=\; \mathbf{e}_1 + \mathbf{e}_2 & (\mathbf{A} - \mathbf{I})^2\, \mathbf{e}_3 \;=\; \mathbf{e}_1 & (\mathbf{A} - \mathbf{I})^3\, \mathbf{e}_3 \;=\; \mathbf{0} \end{array}$$

This means, that $\mathbf{e}_1$ *is an eigenvector,* $\mathbf{e}_2$ *is a generalized eigenvector of degree* 2 *and* $\mathbf{e}_3$ *is a generalized eigenvector of degree* 3*.*

Theorem 2.2 *Let* $\mathbf{A} \in \mathbb{C}^{n \times n}$ *be a complex (or real) matrix with*

$$p_A(\lambda) \;=\; (\lambda_1 - \lambda)^{k_1}\, (\lambda_2 - \lambda)^{k_2} \ldots (\lambda_r - \lambda)^{k_r}.$$

- *Let* λ *be an eigenvalue of* $\mathbf{A}$ *of (algebraic) multiplicity* l*. Then there exist* l *linearly independent generalized eigenvectors (of degree* $\leq l$*). This means:*

  $$\dim\{\, \mathbf{x} \in \mathbb{C}^n \mid (\mathbf{A} - \lambda\mathbf{I})^l \, \mathbf{x} \;=\; \mathbf{0} \,\} \;=\; l.$$

- *Generalized eigenvectors associated to pairwise different eigenvalues of* $\mathbf{A}$ *are linearly independent.*

- *There exists a basis* $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ *of* $\mathbb{C}^n$ *consisting of generalized eigenvectors of* $\mathbf{A}$*. If* $\mathbf{P}$ *is the matrix with this basis as the columns, then*

  $$\mathbf{P}^{-1}\, \mathbf{A}\, \mathbf{P} \;=\; \begin{pmatrix} \boxed{\mathbf{A}_1} & & & \mathbf{0} \\ & \boxed{\mathbf{A}_2} & & \\ & & \ddots & \\ \mathbf{0} & & & \boxed{\mathbf{A}_r} \end{pmatrix}$$

  with $\mathbf{A}_i \in \mathbb{C}^{k_i \times k_i}$ *for all* $i = 1, 2, \ldots, r$*.*

Let us have a look at the case $n = 2$ and $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

1. Characteristic polynomial:

$$\begin{aligned} p_A(\lambda) &= \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\ &= \lambda^2 - \underbrace{(a+d)}_{=:tr(A)}\lambda + \underbrace{ad-bc}_{=\det(A)} = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \end{aligned}$$

with $\lambda_{1,2} = \dfrac{a+d}{2} \pm \sqrt{\dfrac{(a+d)^2}{4} - \det(A)}$.

2. For each λ_i $(i = 1, 2)$ we solve the linear system

$$\begin{pmatrix} a-\lambda_i & b \\ c & d-\lambda_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We have four different cases:

1. $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$

   Example: $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

   We have $p_A(\lambda) = (1-\lambda)^2 - 4 = (\lambda+1)(\lambda-3)$ (two different eigenvalues of algebraic multiplicity 1). A direct calculation shows, that $\dim V(-1) = 1$ and $\dim V(3) = 1$ and the geometric multiplicitiy are (of all eigenvalues) equal to the algebraic multiplicity.

2. $\lambda = \lambda_1 = \lambda_2 \in \mathbb{R}$ with $\dim V(\lambda) = 2$

   Example: $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

   We have $p_A(\lambda) = (2-\lambda)^2$ (one eigenvalue of algebraic multiplicity 2). A direct calculation shows, that $\dim V(2) = 2$ and the geometric multiplicitiy (of the eigenvalue 2) is equal to the algebraic multiplicity.

3. $\lambda = \lambda_1 = \lambda_2 \in \mathbb{R}$ with $\dim V(\lambda) = 1$

   Example: $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

   We have $p_A(\lambda) = (2-\lambda)^2$ (one eigenvalue of algebraic multiplicity 2). A direct calculation shows, that $\dim V(2) = 1$ and the geometric multiplicitiy of the eigenvalue 2 is different of the algebraic multiplicity.

4. $\lambda_2 = \overline{\lambda_1} \in \mathbb{C} - \mathbb{R}$

   Example: $\mathbf{A} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ with $\phi \neq k\pi$

   We have $p_A(\lambda) = (\lambda - \cos\phi)^2 + \sin^2\phi = \lambda^2 - 2\lambda\cos\phi + 1$ with the two different complex zeroes $\lambda_{1,2} = \cos\phi \pm i\sin\phi$.

3 Diagonalization

Let $\mathbf{A}$ and $\mathbf{P}$ be $n \times n$ matrices with $\mathbf{P}$ invertible. Then $\mathbf{A}$ and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ have the same eigenvalues (because they have the same characteristic polynomial).

Definition 3.1 *An $n \times n$ matrix $\mathbf{A}$ is diagonalizable if there is an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} \;=\; \mathbf{D}.$$

Two natural questions:

1. Which square matrices are diagonalizable?
2. If $\mathbf{A}$ is diagonalizable, how do we find the matrix $\mathbf{P}$?

Theorem 3.1 *An $n \times n$ matrix $\mathbf{A}$ is diagonalizable if and only if it has a set of n linearly independent eigenvectors $\mathbf{p}_1, \dots, \mathbf{p}_n$. In this case,*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} \;=\; \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

where $\mathbf{P}$ is the matrix with $\mathbf{p}_1, \dots, \mathbf{p}_n$ as its columns, and $\lambda_1, \dots, \lambda_n$ are the corresponding eigenvalues.

Proof: We prove only one direction of the statement:

$\mathbf{A}$ has n linearly independent eigenvectors $\Longrightarrow$ $\mathbf{A}$ is diagonalizable.

Let $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ be the n linearly independent eigenvectors of $\mathbf{A}$ with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. We form the matrix

$$\mathbf{P} = \begin{pmatrix} | & | & & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_n \\ | & | & & | \end{pmatrix}$$

with the eigenvectors of $\mathbf{A}$ as the columns. Then

$$\mathbf{A}\mathbf{P} \;=\; \begin{pmatrix} | & | & & | \\ \mathbf{A}\mathbf{p}_1 & \mathbf{A}\mathbf{p}_2 & \dots & \mathbf{A}\mathbf{p}_n \\ | & | & & | \end{pmatrix}$$

the column vectors of $\mathbf{A}\mathbf{P}$ are the vectors $\mathbf{A}\mathbf{p}_1, \mathbf{A}\mathbf{p}_2, \dots, \mathbf{A}\mathbf{p}_n$. Using the property of eigenvectors, we get

$$
\begin{aligned}
\mathbf{AP} &= \begin{pmatrix} | & | & & | \\ \mathbf{Ap}_1 & \mathbf{Ap}_2 & \dots & \mathbf{Ap}_n \\ | & | & & | \end{pmatrix} \\
&= \begin{pmatrix} | & | & & | \\ \lambda_1\mathbf{p}_1 & \lambda_2\mathbf{p}_2 & \dots & \lambda_n\mathbf{p}_n \\ | & | & & | \end{pmatrix} \\
&= \begin{pmatrix} | & | & & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \\
&= \mathbf{PD}.
\end{aligned}
$$

where $\mathbf{D}$ is the diagonal matrix with diagonal entries equal to the eigenvalues of $\mathbf{A}$. The matrix $\mathbf{P}$ has maximal rank (and is invertible), because the column vectors are linearly independent. Hence the equation $\mathbf{AP} = \mathbf{PD}$ is equivalent to $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$.

□

Example 3.1 *The matrix* $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ *has the eigenvalues and eigenvectors*

$$
\lambda_1 = 2 \qquad \mathbf{p}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
$$
$$
\lambda_2 = 3 \qquad \mathbf{p}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
$$

Hence $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\mathbf{P}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ *and:*

$$
\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}
$$

Many matrices encountered in economics are (real) symmetric and for these matrices we have the following important result.

Theorem 3.2 (Spectral Theorem for symmetric matrices) *If the real $n \times n$ matrix $\mathbf{A}$ is symmetric ($\mathbf{A} = \mathbf{A}^T$), then:*

1. *All n eigenvalues $\lambda_1, \ldots, \lambda_n$ are real.*
2. *Eigenvectors that correspond to different eigenvalues are orthogonal.*
3. *There exists an orthogonal and real matrix $\mathbf{P}$ ($\mathbf{P}^{-1} = \mathbf{P}^T$) such that*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

The columns $\mathbf{p}_1, \ldots, \mathbf{p}_n$ of the matrix $\mathbf{P}$ are eigenvectors of unit length corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$.

Proof: Let $\mathbf{A}$ be a real and symmetric $n \times n$ matrix.

1. Let $\mathbf{A}\mathbf{p}_i = \lambda_i \mathbf{p}_i$. By complex conjugation of this equation (complex conjugate all entries of the vector and matix, but keep in mind that $\mathbf{A}$ has only real entries) we get

$$\overline{\mathbf{A}\mathbf{p}_i} = \overline{\mathbf{A}}\overline{\mathbf{p}_i} = \mathbf{A}\overline{\mathbf{p}_i} = \overline{\lambda_i}\overline{\mathbf{p}_i}$$

and

$$\lambda_i \mathbf{p}_i^T \overline{\mathbf{p}_i} = (\mathbf{A}\mathbf{p}_i)^T \overline{\mathbf{p}_i} = \mathbf{p}_i^T \mathbf{A}^T \overline{\mathbf{p}_i} = \mathbf{p}_i^T \mathbf{A} \overline{\mathbf{p}_i} = \mathbf{p}_i^T \overline{\lambda_i} \overline{\mathbf{p}_i} = \overline{\lambda_i} \mathbf{p}_i^T \overline{\mathbf{p}_i}$$

Because $\mathbf{p}_i^T \overline{\mathbf{p}_i} = ||\mathbf{p}_i||^2 \neq 0$, we have $\lambda_i = \overline{\lambda_i}$ and λ_i must be a real number.

2. Let $\mathbf{A}\mathbf{p}_i = \lambda_i \mathbf{p}_i$ and $\mathbf{A}\mathbf{p}_j = \lambda_j \mathbf{p}_j$ with $\lambda_i \neq \lambda_j$. Then

$$\begin{aligned} \lambda_i \, \mathbf{p}_i^T \mathbf{p}_j &= (\mathbf{A}\mathbf{p}_i)^T \mathbf{p}_j \\ &= \mathbf{p}_i^T \mathbf{A}^T \mathbf{p}_j \\ &= \mathbf{p}_i^T (\mathbf{A}^T \mathbf{p}_j) \\ &= \mathbf{p}_i^T (\mathbf{A}\mathbf{p}_j) \qquad \text{because } \mathbf{A} = \mathbf{A}^T \\ &= \mathbf{p}_i^T \lambda_j \mathbf{p}_j \\ &= \lambda_j \, \mathbf{p}_i^T \mathbf{p}_j \end{aligned}$$

or

$$\lambda_i \, (\mathbf{p}_i^T \mathbf{p}_j) = \lambda_j \, (\mathbf{p}_i^T \mathbf{p}_j)$$

and because $\lambda_i \neq \lambda_j$, the scalar product of $\mathbf{p}_i$ and $\mathbf{p}_j$ must be zero: $\mathbf{p}_i^T \mathbf{p}_j = \mathbf{p}_i \bullet \mathbf{p}_j = 0$. Hence the two eigenvectors are orthogonal.

3. We give the proof of part 3 only for the case that all eigenvalues $\lambda_1, \ldots, \lambda_n$ are (pairwise) different (and real by part 1). In this case, the corresponding eigenvectors $\mathbf{p}'_1, \ldots, \mathbf{p}'_n$ are orthogonal (by part 2) and hence linearly independent. Now choose for $i = 1, \ldots, n$ an eigenvector of length 1 by

$$\mathbf{p}_i \;:=\; \frac{1}{||\mathbf{p}'_i||}\,\mathbf{p}'_i$$

It is easy to show, that

$$\mathbf{p}_i^T\mathbf{p}_j \;=\; \mathbf{p}_i \bullet \mathbf{p}_j \;=\; \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

The matrix

$$\mathbf{P} = \begin{pmatrix} | & | & & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \\ | & | & & | \end{pmatrix}$$

is an orthogonal matrix, because

$$\mathbf{P}^T\mathbf{P} \;=\; \begin{pmatrix} - & \mathbf{p}_1^T & - \\ - & \mathbf{p}_2^T & - \\ & \cdots & \\ - & \mathbf{p}_n^T & - \end{pmatrix} \begin{pmatrix} | & | & & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{p}_1^T\mathbf{p}_1 & \mathbf{p}_1^T\mathbf{p}_2 & \cdots & \mathbf{p}_1^T\mathbf{p}_n \\ \mathbf{p}_2^T\mathbf{p}_1 & \mathbf{p}_2^T\mathbf{p}_2 & \cdots & \mathbf{p}_2^T\mathbf{p}_n \\ \cdots & \cdots & \ddots & \cdots \\ \mathbf{p}_n^T\mathbf{p}_1 & \mathbf{p}_n^T\mathbf{p}_2 & \cdots & \mathbf{p}_n^T\mathbf{p}_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Hence we have $\mathbf{P}^T = \mathbf{P}^{-1}$

□

Theorem 3.3 (Spectral decomposition of symmetric matrices) *Let* $\mathbf{A}$ *be a symmetric matrix with the set* $\mathbf{p}_1, \ldots, \mathbf{p}_n$ *of orthogonal eigenvectors associated to the real eigenvalues* $\lambda_1, \ldots, \lambda_n$. *Then* $\mathbf{A}$ *can be written as*

$$\boxed{\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{p}_i \mathbf{p}_i^T = \lambda_1 \mathbf{p}_1 \mathbf{p}_1^T + \cdots + \lambda_n \mathbf{p}_n \mathbf{p}_n^T}$$

Proof: For each vector $\mathbf{p}_j$ (of the given ONB) we have

$$\mathbf{A}\mathbf{p}_j = \lambda_j \mathbf{p}_j$$

and

$$\begin{aligned} \left(\sum_{i=1}^{n} \lambda_i \mathbf{p}_i \mathbf{p}_i^T \right) \mathbf{p}_j &= \sum_{i=1}^{n} \lambda_i \mathbf{p}_i \left(\mathbf{p}_i^T \mathbf{p}_j \right) \\ &= \sum_{i=1}^{n} \lambda_i \mathbf{p}_i \delta_{ij} \\ &= \lambda_j \mathbf{p}_j \end{aligned}$$

□

Example 3.2 *The matrix* $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ *is symmetric and has the eigenvalues* $\lambda_1 = -1$ *and* $\lambda_2 = 3$. *The corresponding eigenspaces are*

$$\begin{aligned} V(-1) &= \left\{ t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mid t \in \mathbb{R} \right\} \\ V(3) &= \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\} \end{aligned}$$

The two eigenspaces are orthogonal, because the scalar product of the two spanning vectors is 0. *In order to construct the matrix* $\mathbf{P}$, *we have to use eigenvectors of length* 1 *(unit vectors). A spanning vector of length* 1 *for* $V(-1)$ *is*

$$\mathbf{p}_1 = \frac{1}{\sqrt{1^2 + (-1)^1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and for $V(3)$ *is*

$$\mathbf{p}_2 = \frac{1}{\sqrt{1^2 + 1^1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hence $\mathbf{P} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ *is an orthogonal matrix, because* $\mathbf{P}^{-1} = \mathbf{P}^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ *and*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

Furthermore

$$\begin{aligned} \mathbf{p}_1\mathbf{p}_1^T &= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \\ \mathbf{p}_2\mathbf{p}_2^T &= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \end{aligned}$$

and the spectral decomposition of $\mathbf{A}$ *ist given by*

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = -1 \cdot \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

4 Quadratic forms and matrices

Definition 4.1 *A quadratic form in n variables $\mathbf{x} = (x_1, \ldots, x_n)^T$ is a function of the form*

$$Q_{\mathbf{A}}(\mathbf{x}) \;=\; \sum_{i,j=1}^{n} a_{ij} x_i x_j \;=\; \mathbf{x}^T \mathbf{A} \mathbf{x}$$

where $\mathbf{A} = (a_{ij})$ is an $n \times n$ matrix.

Quadratic forms are important examples of multi-variate functions and $Q_{\mathbf{A}}$ is a homogeneous function of degree 2 in n variables.

Of course, $Q_{\mathbf{A}}(\mathbf{0}) = 0$ for all quadratic forms.

Example 4.1 *$Q(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$ is a quadratic form and can be written as*

$$\begin{aligned}
(x_1\ x_2) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= (x_1\ x_2) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= (x_1\ x_2) \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \ldots..
\end{aligned}$$

Unfortunaetly, there is no unique way to write a given quadratic form in matrix term. But we may resolve this situation by **always choosing A to be symmetric!**

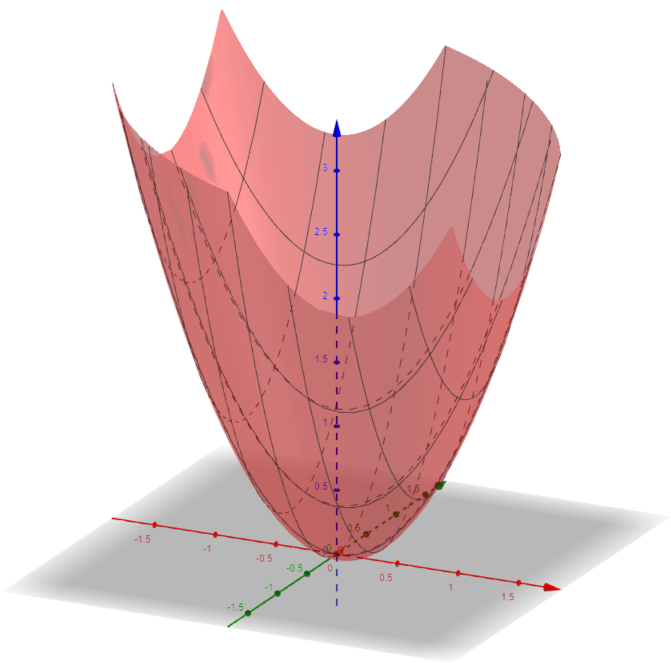
Exercise 4.1 *Let $Q(\mathbf{x}) = \mathbf{x}^T \mathbf{B} \mathbf{x}$ where $\mathbf{B}$ is not symmetric. Let $\mathbf{A} = (\mathbf{B} + \mathbf{B}^T)/2$ and $\mathbf{C} = (\mathbf{B} - \mathbf{B}^T)/2$. Show that $\mathbf{A}$ is symmetric and evaluate both $\mathbf{x}^T \mathbf{A} \mathbf{x}$ and $\mathbf{x}^T \mathbf{C} \mathbf{x}$.*

Example 4.2

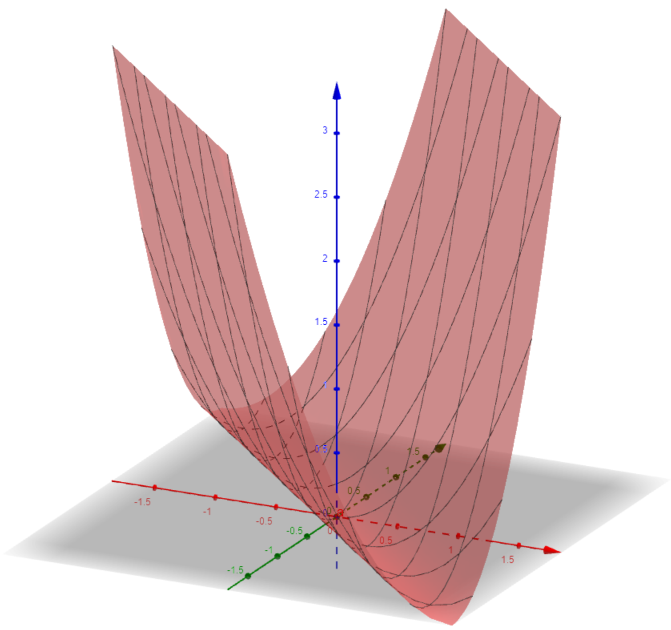
- *The quadratic form* $Q(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$ *can be written as*

$$\left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3}{4}x_2^2.$$

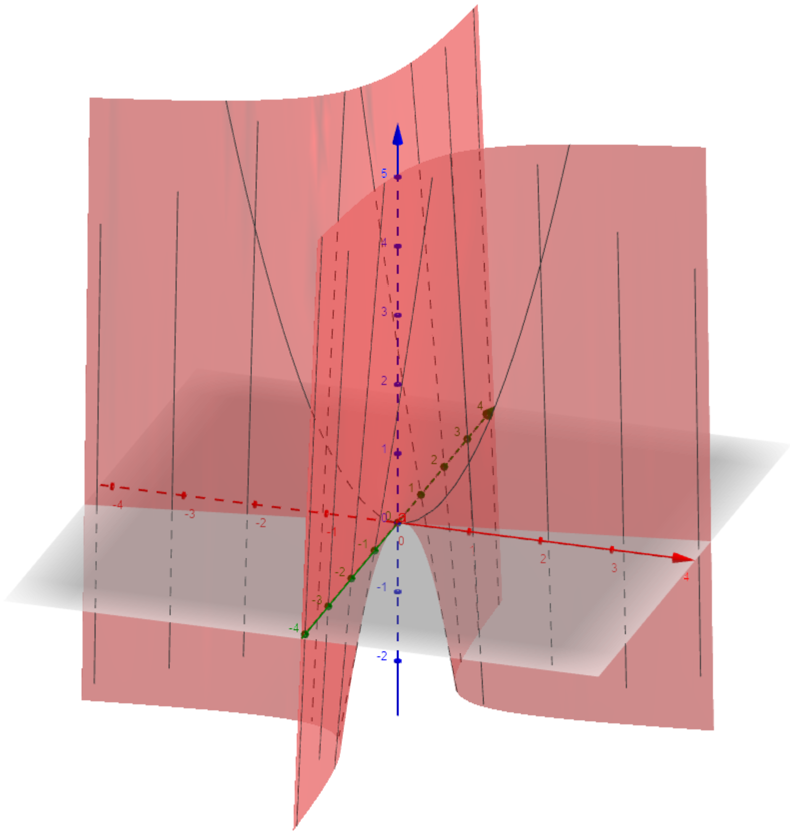
As a sum of squares, it can not be negative and can only be zero when $x_1 + \frac{x_2}{2} = 0$ *and* $x_2 = 0$*, or* $x_1 = x_2 = 0$*. We call this a positive definite quadratic form.*



- *The quadratic form* $Q(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2 \; = \; (x_1 + x_2)^2$ *is always non-negative, but it is zero whenever* $x_1 + x_2 = 0$ *or* $x_1 = -x_2$ *(it is zero for non-zero values of the variables). We call this a positive semi-definite quadratic form.*



- *The quadratic form* $Q(x_1, x_2) = x_1^2 - 6x_1x_2 = (x_1 - 3x_2)^2 - 9x_2^2$ *can be positive or negative. We call this an indefinite quadratic form.*



Definition 4.2 *A quadratic form* $Q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$, *as well as its associated symmetric matrix* $\mathbf{A}$, *is said to be*

$$\begin{array}{rcl} \underline{\textit{positive definite}} & :\Longleftrightarrow & Q_{\mathbf{A}}(\mathbf{x}) > 0 \\ \underline{\textit{positive semi-definite}} & :\Longleftrightarrow & Q_{\mathbf{A}}(\mathbf{x}) \geq 0 \\ \underline{\textit{negative definite}} & :\Longleftrightarrow & Q_{\mathbf{A}}(\mathbf{x}) < 0 \\ \underline{\textit{negative semi-definite}} & :\Longleftrightarrow & Q_{\mathbf{A}}(\mathbf{x}) \leq 0 \end{array}$$

for **all** $\mathbf{x} \neq \mathbf{0}$.

The quadratic form is called <u>*indefinite*</u>, *if there are vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *with* $Q_{\mathbf{A}}(\mathbf{a}) < 0$ *and* $Q_{\mathbf{A}}(\mathbf{b}) > 0$.

It is easy to see, that for $i = 1, \ldots, n$:

$$\boxed{Q_{\mathbf{A}}(\mathbf{e}_i) = a_{ii}.}$$

The technique used in the examples to examine the sign of the quadratic form is known as **completing the squares**. Let us examine the possible signs of a quadratic form $Q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ using the eigenvalues/eigenvectors of the **symmetric** matrix $\mathbf{A}$.

By the **Spectral Theorem for symmetric matrices** we can choose a matrix $\mathbf{P}$ of eigenvectors $\mathbf{p}_1, \ldots, \mathbf{p}_n$ of $\mathbf{A}$, such that $\mathbf{P}^{-1} = \mathbf{P}^T$ and

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{A}$.

Now let $\mathbf{y} := \mathbf{P}^T\mathbf{x}$. This defines new variables $y_1, \ldots, y_n$ as linear combinations of the old ones

$$y_i = \sum_{j=1}^{n} p_{ji}x_j.$$

Further, since $\mathbf{P}\mathbf{P}^T = \mathbf{I}$ we have $\mathbf{x} = \mathbf{P}\mathbf{y}$ and

$$\begin{aligned} Q_{\mathbf{A}}(\mathbf{x}) &= \mathbf{x}^T\mathbf{A}\mathbf{x} \\ &= (\mathbf{P}\mathbf{y})^T\mathbf{A}(\mathbf{P}\mathbf{y}) \\ &= \mathbf{y}^T\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{y} \\ &= \mathbf{y}^T(\mathbf{P}^T\mathbf{A}\mathbf{P})\mathbf{y} \\ &= \mathbf{y}^T \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \mathbf{y} \\ &= \sum_{i=1}^{n} \lambda_i y_i^2. \end{aligned}$$

Thus we completed the squares. The quadratic form is expressed in terms of the new variables as a sum/difference of pure square terms. To determine the sign of the quadratic form, we simply inspect the signs of the eigenvalues of $\mathbf{A}$.

Theorem 4.1 (Sylvester)
If $\mathbf{A}$ is symmetric, then the quadratic form $Q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ is

$$\begin{aligned}
\underline{\textit{positive definite}} &\iff \forall\ \lambda_i > 0\\
\underline{\textit{positive semi-definite}} &\iff \forall\ \lambda_i \geq 0\\
\underline{\textit{negative definite}} &\iff \forall\ \lambda_i < 0\\
\underline{\textit{negative semi-definite}} &\iff \forall\ \lambda_i \leq 0\\
\underline{\textit{indefinite}} &\iff \exists\ \lambda_i > 0 \textit{ and } \lambda_j < 0.
\end{aligned}$$

Checking eigenvalues can be tedious. There is a convenient condition on the matrix $\mathbf{A}$ in terms of certain sub-determinants, which can be used to identify the definiteness of $\mathbf{A}$.

An arbitrary principal minor of order r of an $n \times n$ matrix $\mathbf{A}$ is the determinant of a matrix obtained by deleting $n - r$ rows and $n - r$ columns of $\mathbf{A}$ such that if the ith row (column) is selected then so is the ith column (row). A principal minor is called a leading principal minor of order r if it consists of the first (leading) r rows and columns of $\mathbf{A}$.

Example 4.3 *Let*

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The principal minors of $\mathbf{A}$ *are* $\det(\mathbf{A})$, $\det\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$, $\det\begin{pmatrix} a_{11} & a_{13}\\ a_{31} & a_{33}\end{pmatrix}$, $\det\begin{pmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{pmatrix}$, a_{11}, a_{22} *and* a_{33}.

The leading principal minors are a_{11}, $\det\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$ *and* $\det(\mathbf{A})$.

Theorem 4.2
Let $\mathbf{A}$ be a symmetric $n \times n$ matrix. We denote by D_k the leading principal minor of order k and let Δ_k denote an arbitrary principal minor of order k. Then the quadratic form $Q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ is

$$\begin{aligned}
\underline{\textit{positive definite}} &\iff D_k > 0 \textit{ for } k = 1, \ldots, n\\
\underline{\textit{positive semi-definite}} &\iff \Delta_k \geq 0 \textit{ for all principal minors of order } k = 1, \ldots, n\\
\underline{\textit{negative definite}} &\iff (-1)^k D_k > 0 \textit{ for } k = 1, \ldots, n\\
\underline{\textit{negative semi-definite}} &\iff (-1)^k \Delta_k \geq 0 \textit{ for all principal minors of order } k = 1, \ldots, n.
\end{aligned}$$

Example 4.4

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 13 \end{pmatrix}$$

- *all principal minors of order* 1*:*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 13 \end{pmatrix} \qquad \det(1) \geq 0$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 13 \end{pmatrix} \qquad \det(1) \geq 0$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 13 \end{pmatrix} \qquad \det(13) \geq 0$$

- *all pricipal minors of order* 2

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 13 \end{pmatrix} \qquad \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \geq 0$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 13 \end{pmatrix} \qquad \det\begin{pmatrix} 1 & 3 \\ 3 & 13 \end{pmatrix} = 4 \geq 0$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 13 \end{pmatrix} \qquad \det\begin{pmatrix} 1 & 2 \\ 2 & 13 \end{pmatrix} = 9 \geq 0$$

- *all principal minors of order* 3

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 13 \end{pmatrix} \qquad \det\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 13 \end{pmatrix} = 0 \geq 0$$

Hence **A** *is positive semi-definite and not positive definite.*

Special case: $n = 2$ The quadratic form

$$Q_{\mathbf{A}}(\mathbf{x}) = (x_1\ x_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

- is positive definite if $a_{11} > 0$ and $\det \mathbf{A} = a_{11}a_{22} - a_{12}^2 > 0$;
- is positive semi-definite if $a_{11} \geq 0$, $a_{22} \geq 0$ and $\det \mathbf{A} = a_{11}a_{22} - a_{12}^2 \geq 0$;
- is negative definite if $a_{11} < 0$ and $\det \mathbf{A} = a_{11}a_{22} - a_{12}^2 > 0$;
- is negative semi-definite if $a_{11} \leq 0$, $a_{22} \leq 0$ and $\det \mathbf{A} = a_{11}a_{22} - a_{12}^2 \geq 0$.