Static optimization

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1 Overview about (static) optimization problems

In a general static optimization problem there is

- a real-valued function

$$f(\mathbf{x}) \;=\; f(x_1, \ldots, x_n)$$

  in n variables, the so-called objective function, whose value is to be optimized (maximized or minimized) and

- a set $D \subset \mathbb{R}^n$, the so-called admissible set.

Then the problem is to find (global) maximum or minimum points $\mathbf{x}^* \in D$ of f:

$$\max(\min)\ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D.$$

From now on we will always assume that f is at least 2-times continuously partially differentiable.

Because $\max\ f(\mathbf{x}) \;=\; \min\ -f(\mathbf{x})$ subject to $\mathbf{x} \in D$ we could focus our attention (without loss of generality) on minimizing problems.

Depending on the set D and the function f several different types of optimization problems can arise. At the first level we will distinguish between so-called

1. unconstrained optimization problems:

   D contains no boundary points of D. This means that the set D is an open subset of $\mathbb{R}^n$ and a solution of the optimization problem (if it exists) is an interior point of D.

   Example 1.1 *Solve the following problems or explain why there are no solutions:*

   $\min x^2$ *subject to* $x \in D = (-1, 1)$

   $\min -x^2$ *subject to* $x \in D = (-1, 1)$

   $\min x^2$ *subject to* $x \in D = \mathbb{R}$

   $\min 1/x$ *subject to* $x \in D = (0, 1)$

   $\min -1/x$ *subject to* $x \in D = (0, 1)$

   $\min x^2 - x^4$ *subject to* $x \in D = (-2, 2)$

   $\min x^2 - x^4$ *subject to* $x \in D = (-1, 1)$

   $\min x^2 - x^4$ *subject to* $x \in D = (-0.1, 0.1)$

   $\min \sin(1/x)/x$ *subject to* $x \in D = (0, 1)$

2. constrained optimization problems:

   D contains some boundary points of D. A solution of the optimization problem may be an interior point or a point on the boundary of D.

2 Unconstrained optimization problems

2.1 Local minimizer

Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. Let D be some **open** subset of $\mathbb{R}^n$ and $\mathbf{x}^* \in D$ a local minimizer of f over D. This means that there exists an $\epsilon > 0$ such that for all $\mathbf{x} \in D$ satisfying $||\mathbf{x} - \mathbf{x}^*|| < \epsilon$ we have $f(\mathbf{x}^*) \leq f(\mathbf{x})$.

The term ,,unconstrained" usually refers to the situation where all points $\mathbf{x}$ sufficiently near $\mathbf{x}^*$ are in D. This is automatically true if D is an open set.

We already know:

Theorem 2.1 (First- and second order necessary conditions for optimality)
Suppose that $\nabla^2 f$ is continuous in an open neighbourhood U of $\mathbf{x}^*$ then

$$\mathbf{x}^* \text{ is a local minimizer of } f \implies \nabla f(\mathbf{x}^*) = \mathbf{0} \text{ and } \nabla^2 f(\mathbf{x}^*) \text{ is pos.semidef.}$$

Note that these necessary conditions are not sufficient.

Theorem 2.2 (First- and second order sufficient conditions for optimality)
Suppose that $\nabla^2 f$ is continuous in an open neighbourhood U of $\mathbf{x}^*$ then

$$\nabla f(\mathbf{x}^*) = \mathbf{0} \text{ and } \nabla^2 f(\mathbf{x}^*) \text{ is pos.def.} \implies \mathbf{x}^* \text{ is a (strict) local minimizer of } f$$

Proof:

Because $\nabla^2 f$ is continuous and positive definite at $\mathbf{x}^*$, we can choose an open ball $B = \{\mathbf{x} \mid ||\mathbf{x} - \mathbf{x}^*|| < \epsilon\} \subset D$ where $\nabla^2 f$ remains positive definite. Taking any nonzero vector $\mathbf{v}$ with $||\mathbf{v}|| < \epsilon$, we have $\mathbf{x}^* + \mathbf{v} \in B$ and by Taylor's theorem:

$$\begin{aligned} f(\mathbf{x}^* + \mathbf{v}) &= f(\mathbf{x}^*) + \mathbf{v}^T \nabla f(\mathbf{x}^*) + \frac{1}{2}\,\mathbf{v}^T \nabla^2 f(\mathbf{z})\mathbf{v} \\ &= f(\mathbf{x}^*) + \frac{1}{2}\,\mathbf{v}^T \nabla^2 f(\mathbf{z})\mathbf{v} \end{aligned}$$

for some $\mathbf{z} = \mathbf{x}^* + t \cdot \mathbf{v}$ with $t \in (0, 1)$.

Since $\mathbf{z} = \mathbf{x}^* + t \cdot \mathbf{v} \in B$, we have $\mathbf{v}^T \nabla^2 f(\mathbf{z})\mathbf{v} > 0$ and therefore $f(\mathbf{x}^* + \mathbf{v}) > f(\mathbf{x}^*)$. □

2.2 Global minimizer

Of course, all local minimizers of a function f are candidates for global minimizing, but obviously, an arbitrary function may not realise a global minimum in an open set D. For instance, look at $f(x) = -x^2$ subject to $x \in D = (-1, 1)$.

There are only general results in the case where f is a convex function on D. Because we define convexity of the function f by the inequality

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \quad \leq \quad tf(\mathbf{x}) + (1-t)f(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in D$ and all $t \in [0,1]$, all points $t\mathbf{x} + (1-t)\mathbf{y}$ (points between $\mathbf{x}$ and $\mathbf{y}$) should lie in D. Hence D must be a convex set.

Theorem 2.3 *Let f be a convex (resp. concave) and differentiable function on the convex (and open) set D. Then*

$$\boxed{\mathbf{x}^* \textit{ is a global minimizer (resp. maximizer) of } f \iff \nabla f(\mathbf{x}^*) \;=\; \mathbf{0}}$$

Proof (for convex f):

- „$\Longrightarrow$"
  Clear!?
- „$\Longleftarrow$"
  Let $\nabla f(\mathbf{x}^*) \;=\; \mathbf{0}$ and suppose that $\mathbf{x}^*$ is **not** a global minimizer of f on D. Then we can find a point $\mathbf{y} \in D$ with $f(\mathbf{y}) < f(\mathbf{x}^*)$.
  Consider the line segment that joins $\mathbf{x}^*$ to $\mathbf{y}$, that is

$$\mathbf{z} \;=\; \mathbf{z}(t) \quad = \quad t\mathbf{y} + (1-t)\mathbf{x}^* \;=\; \mathbf{x}^* + t(\mathbf{y} - \mathbf{x}^*)$$

for all $t \in [0,1]$. Of course, $\mathbf{z} \in D$ because D is a convex set. Hence

$$\begin{aligned} \nabla f(\mathbf{x}^*)^T\,(\mathbf{y} - \mathbf{x}^*) \quad &= \quad \frac{d}{dt} f(\mathbf{x}^* + t(\mathbf{y} - \mathbf{x}^*))\Big|_{t=0} \\ &= \quad \lim_{t \to 0+} \frac{f(\mathbf{x}^* + t(\mathbf{y} - \mathbf{x}^*)) - f(\mathbf{x}^*)}{t} \\ &\leq \quad \lim_{t \to 0+} \frac{tf(\mathbf{y}) + (1-t)f(\mathbf{x}^*) - f(\mathbf{x}^*)}{t} \\ &= \quad \lim_{t \to 0+} \frac{t(f(\mathbf{y}) - f(\mathbf{x}^*))}{t} \\ &= \quad f(\mathbf{y}) - f(\mathbf{x}^*) \;<\; 0. \end{aligned}$$

Therefore, $\nabla f(\mathbf{x}^*) \;\neq\; \mathbf{0}$! Contradiction.
Hence, $\mathbf{x}^*$ is a global minimizer of f on D.

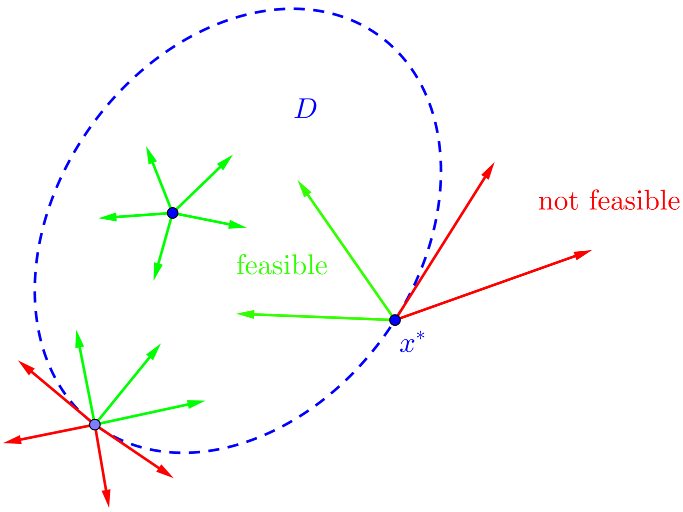
□

3 Constrained optimization problems

3.1 General remarks

In the previous case we have used the fact that for every direction $\mathbf{v}$ points of the form $\mathbf{x}^* + t\mathbf{v}$ belong to D (for sufficiently small t). This is no longer true if D has a boundary and $\mathbf{x}^*$ is a point on this boundary.

Definition 3.1 *Let $D \subset \mathbb{R}^n$ and $\mathbf{x}^* \in D$. A vector $\mathbf{v} \in \mathbb{R}^n$ is called a feasible direction in $\mathbf{x}^*$ if $\mathbf{x}^* + t\mathbf{v} \in D$ for all t with $0 \le t < t_0$.*



If not all directions $\mathbf{v}$ are feasible in $\mathbf{x}^*$, then the condition $\nabla f(\mathbf{x}^*) = \mathbf{0}$ is no longer necessary for local optimality. But we can prove the following result.

Theorem 3.1 *If $\mathbf{x}^*$ is a local minimum of the continuously differentiable function f on D, then*

$$\partial_{\mathbf{v}} f(\mathbf{x}^*) = \nabla f(\mathbf{x}^*)^T \mathbf{v} \geq 0$$

for every feasible direction $\mathbf{v}$ and

$$\mathbf{v}^T \nabla^2 f(\mathbf{x}^*) \mathbf{v} \geq 0$$

for all feasible directions with $\partial_{\mathbf{v}} f(\mathbf{x}^*) = 0$.

There are two cases:

1. $\partial D \not\subset D$

   There are boundary points of D which are not elements of D. This case is too difficult and we need a specific method, adapted to the concrete set D, to solve the optimization problem. We will **not** follow up on this type of problem.

2. $\partial D \subset D$

   The complete boundary ∂D of D is in D; this means that D is closed.

From now on let D always be closed.

We recall the following basic existence result for **closed and bounded** sets D:

Theorem 3.2 (Weierstrass-Theorem) *If f is a continuous function and D is a closed and bounded set then there exists a global minimum of f over D.*

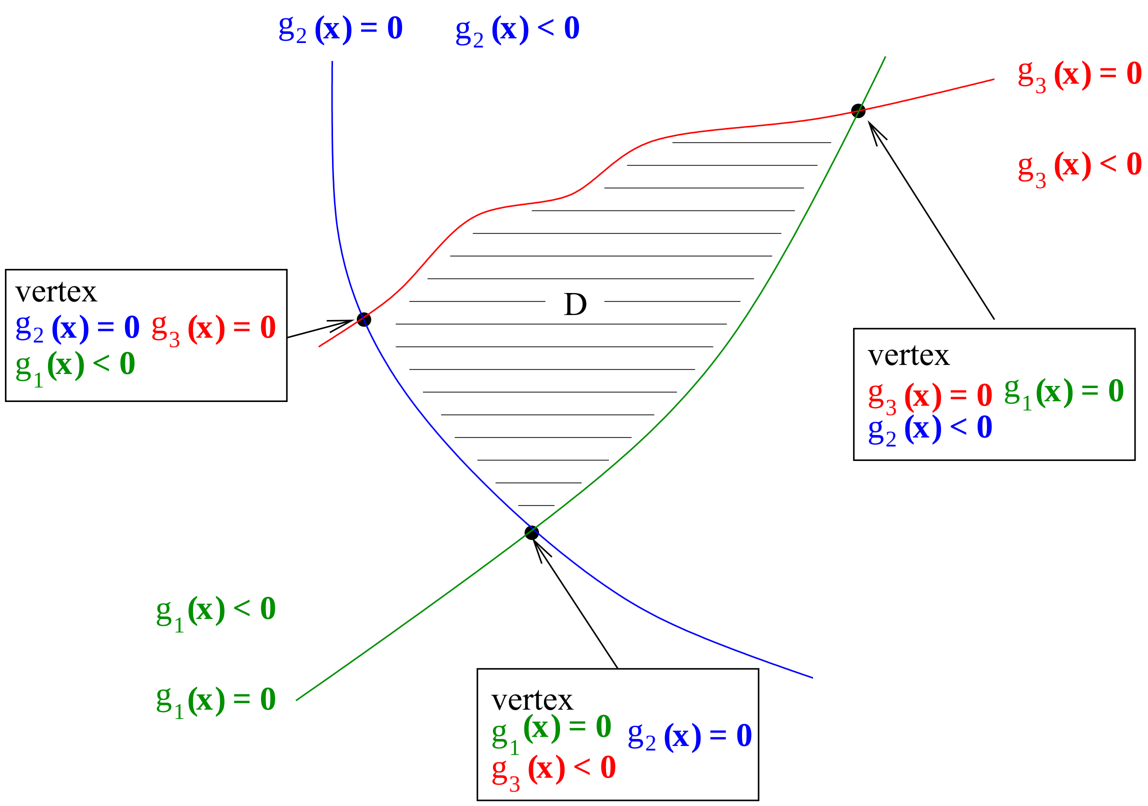
(General) Algorithm for finding a global minimum

1. Find all interior points of D satisfying $\nabla f(\mathbf{x}^*) = \mathbf{0}$ (stationary points).
2. Find all points where ∇f does not exist (critical points).
3. Find all boundary points satisfying $\partial_{\mathbf{v}} f(\mathbf{x}^*) \geq 0$ for all feasible directions $\mathbf{v}$.
4. Compare all values at all these candidate points and choose one smallest one.

In almost all interesting optimization problems the admissible set D is given by a set of inequalities (or equations):

$$D = \{\mathbf{x} \in \mathbb{R}^n \mid g_1(\mathbf{x}) \leq c_1, g_2(\mathbf{x}) \leq c_2, \ldots, g_m(\mathbf{x}) \leq c_m\} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{g}(\mathbf{x}) \leq \mathbf{c}\}$$

with $\mathbf{g} = (g_1, \ldots, g_m)^T$, $g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{c} = (c_1, \ldots, c_m)^T$.



It is easy to see that one equation of the form $g(\mathbf{x}) = c$ can be expressd by the two inequalities $g(\mathbf{x}) \leq c$ and $-g(\mathbf{x}) \leq -c$. Hence all sets described by a set of equations could be described by a set of inequalites and it would be enough to study sets described by inequalities.

But for practical reasons we will discuss the two cases separately.

Definition 3.2 *For the optimization problem*

$$\max(\min) \qquad y = f(x_1, x_2, \ldots, x_n) = f(\mathbf{x})$$

$$\textit{subject to} \qquad \begin{cases} g_1(x_1, x_2, \ldots, x_n) = g_1(\mathbf{x}) \leq c_1 \\ g_2(x_1, x_2, \ldots, x_n) = g_2(\mathbf{x}) \leq c_2 \\ \cdots \\ g_m(x_1, x_2, \ldots, x_n) = g_m(\mathbf{x}) \leq c_m \end{cases}$$

the function (in $n + m$ variables)

$$L(x_1, \cdots, x_n, \lambda_1, \ldots, \lambda_m) \;=\; f(x_1, x_2, \cdots, x_n) - \sum_{j=1}^{m} \lambda_j \left(g_j(x_1, x_2, \cdots, x_n) - c_j\right)$$

shortly

$$L(\mathbf{x}, \boldsymbol{\lambda}) \;=\; f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j \left(g_j(\mathbf{x}) - c_j\right) \;=\; f(\mathbf{x}) - \boldsymbol{\lambda}^T(\mathbf{g}(\mathbf{x}) - \mathbf{c})$$

is called Lagrange function of the optimization problem.

3.2 $D = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{g}(\mathbf{x}) = \mathbf{c}\}$

3.2.1 The two-variable case

A (free) maximum of $f(x_1, x_2)$ is a mountain top on the graph of f; the constrained maximum is the highest point on a path along the graph. This path lies directly over the path in the domain of f, given by the constraint $g(x_1, x_2) = c$.

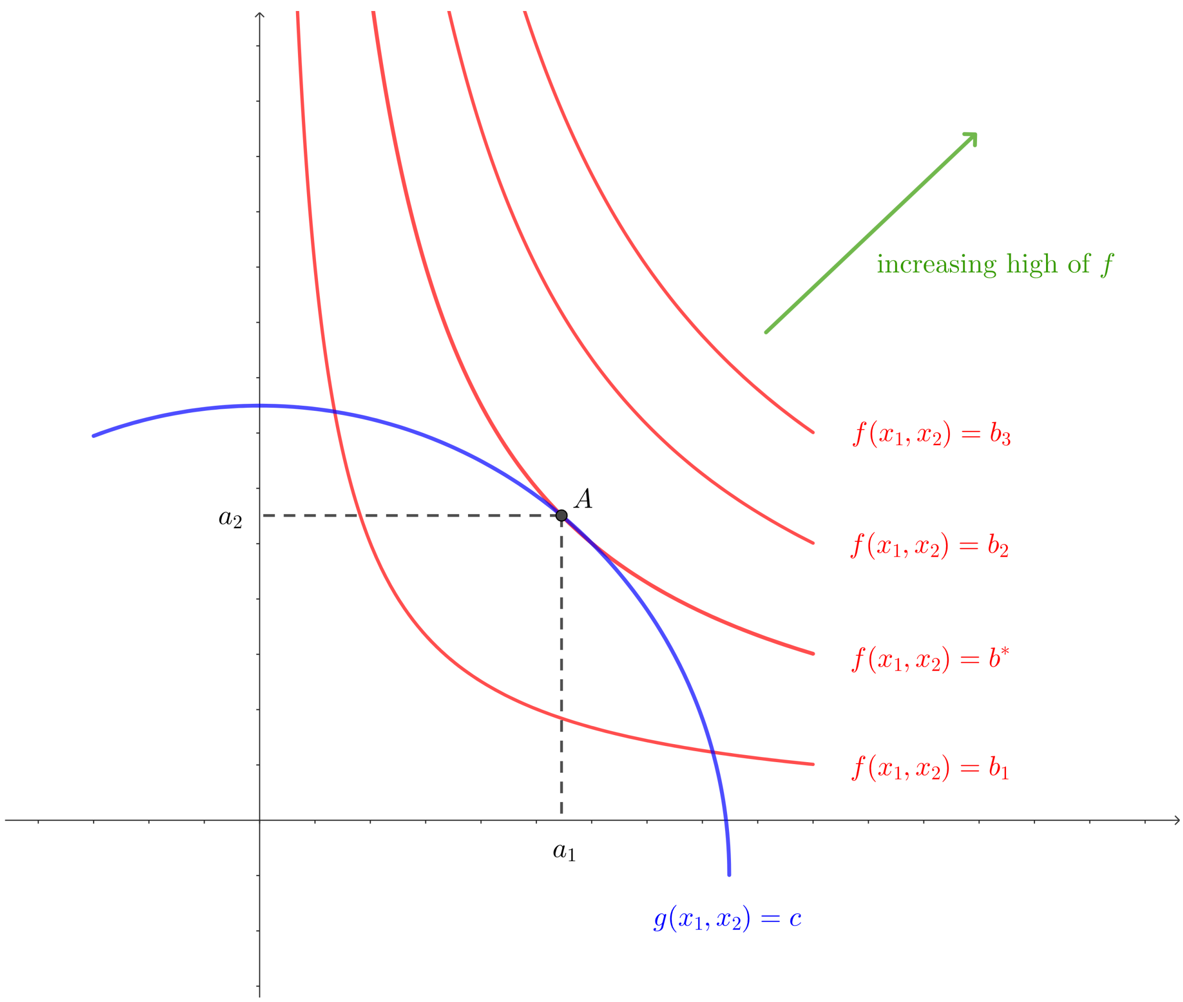
The constraint $g(x_1, x_2) = c$ is simply the contour line (level set) of the function g associated the hight c. We try to solve the following optimization problem:

$$\begin{aligned} \max \quad & y = f(x_1, x_2) = f(\mathbf{x}) \\ \text{subject to} \quad & g(x_1, x_2) = c \end{aligned}$$

Suppose now, for simplicity, that

- f is an increasing function ($f_{x_1}, f_{x_2} > 0$) and strictly quasi-concave and
- g is strictly quasi-convex.

Than the contour lines of f and the constraint $g(x_1, x_2) = c$ are as shown in the following figure:



We see, in this case we have an unique (because f is strictly quasi-concave and g strictly quasi-convex) solution of the maximization problem at the point A. Generally, constrained maxima/minima may not exist, or be unique.

Assuming that there exists a unique constrained maxima of f. If we have a look at the figure, we may see that **at the point A, the slope of the f-contour line $f_{x_1}, f_{x_2} = b^*$ and the slope of the constraint $g(x_1, x_2) = c$ are the same!**

Proof: Suppose, that there is a local solution $x_2 = h(x_1)$ of $g(x_1, x_2) = c$ near A, so $g(x_1, h(x_1)) = c$. Hene, for all points at the contour line $g(x_1, h(x_1)) = c$ near A we have:

$$f(x_1, x_2) \;=\; f(x_1, h(x_1)) \;=:\; F(x_1).$$

Because the point $A = (a_1, a_2)$ is a local maximum of F (for all x_1 near a_1), we have the necessary condition

$$\begin{aligned} 0 \;=\; F'(x_1)\,\big|_{x_1=a_1} &= f_{x_1}(x_1, h(x_1)) + f_{x_2}(x_1, h(x_1)) \cdot h'(x_1)\,\big|_{x_1=a_1} \\ &= f_{x_1}(a_1, a_2) + f_{x_2}(a_1, a_2) \cdot h'(a_1) \end{aligned}$$

or

$$h'(a_1) \;=\; -\frac{f_{x_1}(a_1, a_2)}{f_{x_2}(a_1, a_2)}$$

Otherwise, if we differiantiate the equation $g(x_1, h(x_1)) = c$ with respect to x_1, we get

$$0 \;=\; g_{x_1}(x_1, h(x_1)) + g_{x_2}(x_1, h(x_1)) \cdot h'(x_1)$$

and

$$h'(a_1) \;=\; -\frac{g_{x_1}(a_1, a_2)}{g_{x_2}(a_1, a_2)}$$

□

By implicit differentiation we can express this property as

$$-\frac{f_{x_2}(a_1, a_2)}{f_{x_1}(a_1, a_2)} \;=\; -\frac{g_{x_2}(a_1, a_2)}{g_{x_1}(a_1, a_2)} \qquad \text{(same slope at } A\text{)}$$

or

$$\frac{f_{x_1}(a_1, a_2)}{g_{x_1}(a_1, a_2)} \;=\; \frac{f_{x_2}(a_1, a_2)}{g_{x_2}(a_1, a_2)} \;=\; \lambda \qquad \underline{\text{Lagrange-multiplier}}$$

This equation can be spitted in two equations:

$$\begin{aligned} f_{x_1}(a_1, a_2) &= \lambda\, g_{x_1}(a_1, a_2) \\ f_{x_2}(a_1, a_2) &= \lambda\, g_{x_2}(a_1, a_2) \end{aligned}$$

This means, to find A we have to find all solutions (x_1, x_2, λ) of the following system of three equations:

$$\begin{aligned} f_{x_1}(x_1, x_2) &= \lambda\, g_{x_1}(x_1, x_2) \\ f_{x_2}(x_1, x_2) &= \lambda\, g_{x_2}(x_1, x_2) \\ g(x_1, x_2) &= c \end{aligned}$$

3.2.2 The general case

Given the following optimization problem:

$$\max(\min) \qquad y = f(x_1, x_2, \ldots, x_n) = f(\mathbf{x})$$

$$\text{subject to} \qquad \begin{cases} g_1(x_1, x_2, \ldots, x_n) = g_1(\mathbf{x}) = c_1 \\ g_2(x_1, x_2, \ldots, x_n) = g_2(\mathbf{x}) = c_2 \\ \cdots \\ g_m(x_1, x_2, \ldots, x_n) = g_m(\mathbf{x}) = c_m \end{cases}$$

Theorem 3.3 *Suppose that*

- $f, g_1, \ldots, g_m$ *are defined on a set* $S \subset \mathbb{R}^n$
- $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ *is an interior point of* S *that solves the optimization problem*
- $f, g_1, \ldots, g_m$ *are continuously partial differentiable in a ball around* $\mathbf{x}^*$
- *the Jacobi-matrix of the constraint functions*

$$D\mathbf{g}(\mathbf{x}) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}) & \frac{\partial g_1}{\partial x_2}(\mathbf{x}) & \ldots & \frac{\partial g_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1}(\mathbf{x}) & \frac{\partial g_m}{\partial x_2}(\mathbf{x}) & \ldots & \frac{\partial g_m}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

has rank m *in* $\mathbf{x} = \mathbf{x}^*$.

Necessary condition

Then there exist unique numbers $\lambda_1^*, \ldots, \lambda_m^*$ *such that* $(\mathbf{x}^*, \boldsymbol{\lambda}^*) = (x_1^*, x_2^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*)$ *is a stationary point of the Lagrange-function:*

$$L_{x_1}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0, \ldots, L_{x_n}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0$$

and   *shortly* $\boxed{\nabla L(\mathbf{x}^*, \boldsymbol{\lambda}^*) = \mathbf{0}}$

$$L_{\lambda_1}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0, \ldots, L_{\lambda_m}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0$$

or expanded

$$\nabla f(\mathbf{x}^*) - \sum_{j=1}^{m} \lambda_j^* \, \nabla g_j(\mathbf{x}^*) = \mathbf{0} \qquad (\star)$$

Sufficient condition

If there exist numbers $\lambda_1^*, \ldots, \lambda_m^*$ *and an admissible* $\mathbf{x}^*$ *which together satisfy the necessary condition, and if the Lagrange function* L *is concave (convex) in* $\mathbf{x}$ *and* S *is convex, then* $\mathbf{x}^*$ *solves the maximization (minimization) problem.*

Remark:

The condition that $D\mathbf{g}(\mathbf{x}^*)$ has rank m means, that the gradients $\nabla g_1(\mathbf{x}^*), \ldots, \nabla g_m(\mathbf{x}^*)$ (the rows of $D\mathbf{g}(\mathbf{x}^*)$) are linearly independent. Equation ($\star$) can be written as

$$\nabla f(\mathbf{x}^*) \;=\; \sum_{j=1}^{m} \lambda_j^* \, \nabla g_j(\mathbf{x}^*).$$

This means that in the point $\mathbf{x}^*$ (solution of the optimization problem) the gradient of f is a linear combination of the gradients of all constraint functions.

Proof:

Necessary condition We get a nice argument for condition ($\star$) by studying the optimal value function

$$f^*(\mathbf{c}) \;=\; \max\{f(\mathbf{x}) \mid \mathbf{g}(\mathbf{x}) = \mathbf{c}\}$$

If f is a profit function and $\mathbf{c} = (c_1, \ldots, c_m)$ denotes a resource vector, then $f^*(\mathbf{c})$ is the maximum profit obtainable given the available resource vector $\mathbf{c}$.

In the following argument we **assume that $f^*(\mathbf{c})$ is differentiable.**

Fix a vector $\mathbf{c}^*$ and let $\mathbf{x}^*$ be the corresponding optimal solution. Then $f(\mathbf{x}^*) = f^*(\mathbf{c}^*)$ and obviously for all $\mathbf{x}$ we have $f(\mathbf{x}) \leq f^*(\mathbf{g}(\mathbf{x}))$.

Hence

$$\phi(\mathbf{x}) \;:=\; f(\mathbf{x}) - f^*(\mathbf{g}(\mathbf{x})) \;\leq\; 0$$

has a maximum in $\mathbf{x} = \mathbf{x}^*$, so

$$0 \;=\; \frac{\partial \phi}{\partial x_i}(\mathbf{x}^*) \;=\; \frac{\partial f}{\partial x_i}(\mathbf{x}^*) - \sum_{j=1}^{m} \left[\frac{\partial f^*}{\partial c_j}(\mathbf{c})\right]_{\mathbf{c}=\mathbf{g}(\mathbf{x}^*)} \frac{\partial g_j}{\partial x_i}(\mathbf{x}^*)$$

Define

$$\lambda_j^*(\mathbf{c}) \;:=\; \frac{\partial f^*}{\partial c_j}(\mathbf{c}) \;\approx\; f^*(\mathbf{c} + \mathbf{e}_j) - f^*(\mathbf{c})$$

and equation ($\star$) follows.

Sufficient condition Suppose that $L = L(\mathbf{x})$ is a concave (resp. convex) function in the variable $\mathbf{x}$. The necessary condition means that $\mathbf{x}^*$ is a stationary point of L, this means $\nabla_{\mathbf{x}} L(\mathbf{x}^*) = \mathbf{0}$. Then by Theorem 2.3 we know that $\mathbf{x}^*$ is a global maximizer (resp. minimizer) of L and this means that

$$\begin{aligned} L(\mathbf{x}^*) \;&=\; f(\mathbf{x}^*) - \sum_{j=1}^{m} \lambda_j^*(g_j(\mathbf{x}^*) - c_j) \\ &\geq\; f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j^*(g_j(\mathbf{x}) - c_j) \\ &=\; L(\mathbf{x}) \end{aligned}$$

for all $\mathbf{x} \in S$. But for all admissible $\mathbf{x}$ we have $g_j(\mathbf{x}) = c_j$. Hence $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all admissible $\mathbf{x} \in S$. □

The equation

$$\begin{aligned}\lambda_j^*(\mathbf{c}) &= \frac{\partial f^*}{\partial c_j}(\mathbf{c}) \\ &\approx f^*(\mathbf{c}+\mathbf{e}_j) - f^*(\mathbf{c}) = f^*(c_1,\ldots,c_j+1,\ldots,c_m) - f^*(c_1,\ldots,c_j,\ldots,c_m)\end{aligned}$$

tells us, that the Lagrange multiplicator $\lambda_j^*(\mathbf{c})$ for the jth constraint is the rate at which the optimal value of the objective function changes with respect to the changes in the constant c_j.

Suppose that $f^*(\mathbf{c})$ is the maximum profit that a firm can obtain from a production process when $c_1,\ldots,c_m$ are the available quantities of m different resources. Then $\lambda_j^*(\mathbf{c})$ is the marginal profit that a firm can earn per extra unit of resource j, and therefore the firm's marginal willingness to pay for this resource. If the firm could pay more of this resource at a price below $\lambda_j^*(\mathbf{c})$ per unit, it could earn more profit by doing so. But if the price exceeds $\lambda_j^*(\mathbf{c})$ per unit, the firm could increase its profit by selling a small quantity of this resource at this price.

In economics, the number $\lambda_j^*(\mathbf{c})$ is referred to a so called shadow price of the resource j.

Example 3.1 *Given the following optimization problem:*

$$\begin{array}{rl} \max & f(x_1,x_2) = x_1^{\alpha} x_2^{\beta} \\ \textit{subject to} & g(x_1,x_2) = p_1x_1 + p_2x_2 = c \end{array}$$

The necessary condition ($\star$) will only work, if the optimization problem meets the requirements from Theorem 3.3. We will check it.

- *We take $S = \mathbb{R}^2_{++}$, $x_1, x_2 > 0$ (obviously, a solution of the maximization problem does not lie on the boundary of $\mathbb{R}^2_{++}$).*
- *Hence a solution should be an interior point of S.*
- *The functions f and g are continously partially differentiable in S.*
- *The Jacobi-matrix of g (the gradient) is*

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

and has the maximal rank ($= 1$) for all $(x_1, x_2) \in S$, if $(p_1, p_2) \neq (0,0)$. Think (shortly) about the solution of the optimization problem in the case $(p_1, p_2) = (0,0)$.

Hence we are allowed to use the criterion ($\star$) to find a solution. Step by step we get:

- $L(x_1,x_2,\lambda) \;=\; x_1^{\alpha}x_2^{\beta} - \lambda(p_1x_1 + p_2x_2 - c)$
- $\nabla L(x_1,x_2,\lambda) \;=\; \nabla f(x_1,x_2) - \lambda \nabla g(x_1,x_2) \;=\; \begin{pmatrix} \alpha x_1^{\alpha-1}x_2^{\beta} - \lambda p_1 \\ \beta x_1^{\alpha}x_2^{\beta-1} - \lambda p_2 \\ -(p_1x_1 + p_2x_2 - c) \end{pmatrix}$
- $\begin{pmatrix} \alpha x_1^{\alpha-1}x_2^{\beta} - \lambda p_1 \\ \beta x_1^{\alpha}x_2^{\beta-1} - \lambda p_2 \\ -(p_1x_1 + p_2x_2 - c) \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ *or*

$$\begin{array}{lrcl} \textit{E1:} & \alpha x_1^{\alpha-1}x_2^{\beta} & = & \lambda p_1 \\ \textit{E2:} & \beta x_1^{\alpha}x_2^{\beta-1} & = & \lambda p_2 \\ \textit{E3:} & p_1x_1 + p_2x_2 & = & c \end{array}$$

- *E1/E2*

$$\frac{\alpha x_1^{\alpha-1}x_2^{\beta}}{\beta x_1^{\alpha}x_2^{\beta-1}} \;=\; \frac{\lambda p_1}{\lambda p_2} \;\Leftrightarrow\; \frac{\alpha x_2}{\beta x_1} \;=\; \frac{p_1}{p_2} \;\Leftrightarrow\; x_2 \;=\; \frac{p_1}{p_2}\frac{\beta}{\alpha}x_1$$

- *x_2 in E3*

$$p_1x_1 + p_2x_2 \;=\; c \;\Leftrightarrow\; p_1x_1 + p_2\left(\frac{p_1}{p_2}\frac{\beta}{\alpha}x_1\right) \;=\; c \;\Leftrightarrow\; x_1^* \;=\; \frac{c\alpha}{p_1(\alpha+\beta)}$$

- *x_1 in x_2*

$$x_2^* = \frac{p_1}{p_2}\frac{\beta}{\alpha}x_1 = \frac{p_1}{p_2}\frac{\beta}{\alpha}\frac{c\alpha}{p_1(\alpha+\beta)} = \frac{c\beta}{p_2(\alpha+\beta)}$$

- *x_1^* and x_2^* in E1*

$$\lambda^* = \frac{\alpha\left(\frac{c\alpha}{p_1(\alpha+\beta)}\right)^{\alpha-1}\left(\frac{c\beta}{p_2(\alpha+\beta)}\right)^{\beta}}{p_1} = \frac{\alpha^\alpha\beta^\beta c^{\alpha+\beta-1}}{p_1^\alpha p_2^\beta(\alpha+\beta)^{\alpha+\beta-1}}$$

- *The optimal value function of the problem is*

$$\begin{aligned} f^*(c) &= \max\{f(x_1,x_2) \mid g(x_1,x_2)=c\} \\ &= (x_1^*)^\alpha(x_2^*)^\beta \\ &= \left(\frac{c\alpha}{p_1(\alpha+\beta)}\right)^{\alpha}\left(\frac{c\beta}{p_2(\alpha+\beta)}\right)^{\alpha} \\ &= \frac{\alpha^\alpha\beta^\beta}{p_1^\alpha p_2^\beta(\alpha+\beta)^{\alpha+\beta}}\,c^{\alpha+\beta} \end{aligned}$$

A direct calculation confirms $\frac{\partial f^*}{\partial c}(c) = \lambda^*$.

- *Hesse matrix of L with respect to* $\mathbf{x}$

$$\nabla^2_{\mathbf{x}}L(\mathbf{x}) = \begin{pmatrix} \alpha(\alpha-1)x_1^{\alpha-2}x_2^\beta & \alpha\beta x_1^{\alpha-1}x_2^{\beta-1} \\ \alpha\beta x_1^{\alpha-1}x_2^{\beta-1} & \beta(\beta-1)x_1^\alpha x_2^{\beta-2} \end{pmatrix}$$

- *If $\nabla^2_{\mathbf{x}}L(\mathbf{x})$ is negative definite (for all $x_1, x_2 > 0$) then L is concave and $\mathbf{x}^* = (x_1^*, x_2^*)$ solves the maximization problem. Is $\nabla^2_{\mathbf{x}}L(\mathbf{x})$ negative definite?*

  We know that $\nabla^2_{\mathbf{x}}L(\mathbf{x})$ is negative semi-definite if and only if

$$\begin{aligned} \alpha(\alpha-1)\ \underbrace{x_1^{\alpha-2}x_2^\beta}_{>0\ if\ x_1,x_2>0} &\leq 0 \\ \beta(\beta-1)\ \underbrace{x_1^{\alpha}x_2^{\beta-2}}_{>0\ if\ x_1,x_2>0} &\leq 0 \end{aligned}$$

  and

$$\begin{aligned} \det\nabla^2_{\mathbf{x}}L(\mathbf{x}) &= \alpha(\alpha-1)x_1^{\alpha-2}x_2^\beta\beta(\beta-1)x_1^\alpha x_2^{\beta-2} - \alpha\beta x_1^{\alpha-1}x_2^{\beta-1}\alpha\beta x_1^{\alpha-1}x_2^{\beta-1} \\ &= \alpha\beta(1-\alpha-\beta)\,\underbrace{x_1^{2\alpha-2}x_2^{2\beta-2}}_{>0\ if\ x_1,x_2>0} \\ &\geq 0. \end{aligned}$$

Hence

$$\begin{aligned} \alpha(\alpha-1) &\leq 0 \\ \beta(\beta-1) &\leq 0 \\ \alpha\beta(1-\alpha-\beta) &\geq 0 \end{aligned}$$

and the combination of these three relations gives the following result:

$$\nabla^2_{\mathbf{x}} L(\mathbf{x}) \text{ is negative semi-definite} \iff 0 \leq \alpha, \beta \leq 1 \text{ and } 1 \geq \alpha + \beta.$$

Exercise 3.1 *Solve the following optimization problem*

$$\begin{array}{ll} \max & f(x_1, x_2) = a\ln(x_1) + b\ln(x_2) \\ \textit{subject to} & g(x_1, x_2) = p_1 x_1 + p_2 x_2 = c \end{array}$$

Compare the solution to that obtained in the above example.

3.3 $D = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{g}(\mathbf{x}) \leq \mathbf{c}\}$

Given the following optimization problem:

$$\max \quad y = f(x_1, x_2, \dots, x_n) = f(\mathbf{x})$$

$$\text{subject to} \quad \begin{cases} g_1(x_1, x_2, \dots, x_n) = g_1(\mathbf{x}) \leq c_1 \\ g_2(x_1, x_2, \dots, x_n) = g_2(\mathbf{x}) \leq c_2 \\ \dots \\ g_m(x_1, x_2, \dots, x_n) = g_m(\mathbf{x}) \leq c_m \end{cases}$$

Definition 3.3 *Let $\mathbf{x}^*$ be the solution of the maximization problem. The constraint $g_i(\mathbf{x}) \leq c_i$ is called*

- *binding (or active) at $\mathbf{x}^*$, if $g_i(\mathbf{x}^*) = c_i$ and*
- *not binding (or inactive) at $\mathbf{x}^*$, if $g_i(\mathbf{x}^*) < c_i$.*

Theorem 3.4 *Suppose that*

- *$f, g_1, \dots, g_m$ are defined on a set $S \subset \mathbb{R}^n$*
- *$\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ is an interior point of S that solves the maximization problem*
- *$f, g_1, \dots, g_m$ are continuously partially differentiable in a ball around $\mathbf{x}^*$*
- *the constraints are ordered in such a way, that the first m_0 constraints are binding at $\mathbf{x}^*$ and all the remaining $m - m_0$ constraints are not binding,*
- *the Jacobi-matrix of the binding constraint functions*

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial g_1}{\partial x_n}(\mathbf{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{m_0}}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial g_{m_0}}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

  has rank m_0 in $\mathbf{x} = \mathbf{x}^*$.

Necessary condition

Then there exist unique real numbers $\boldsymbol{\lambda}^* = (\lambda_1^*, \dots, \lambda_m^*)$ such that

1. *$L_{x_1}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0, \dots, L_{x_n}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0$,*
2. *$\lambda_1^* \geq 0, \dots, \lambda_m^* \geq 0$,*
3. *$\lambda_1^* \cdot [g_1(\mathbf{x}^*) - c_1] = 0, \dots, \lambda_m^* \cdot [g_m(\mathbf{x}^*) - c_m] = 0$ and*
4. *$g_1(\mathbf{x}^*) \leq c_1, \dots, g_m(\mathbf{x}^*) \leq c_m$.*

Conditions 1., 2. and 3. are often called Karush-Kuhn-Tucker-conditions.

Proof:

Necessary condition We study the optimal value function

$$f^*(\mathbf{c}) \;=\; \max\{f(\mathbf{x}) \mid \mathbf{g}(\mathbf{x}) \le \mathbf{c}\}$$

This value function must be nondecreasing in each variable $c_1, \ldots, c_m$. This is because as c_j increases with all other variables held fixed, the admissible set becomes larger; hence $f^*(\mathbf{c})$ can not decrease.

In the following argument we **assume that** $f^*(\mathbf{c})$ **is differentiable.**

Fix a vector $\mathbf{c}^*$ and let $\mathbf{x}^*$ be the corresponding optimal solution. Then $f(\mathbf{x}^*) = f^*(\mathbf{c}^*)$. For any $\mathbf{x}$ we have $f(\mathbf{x}) \le f^*(\mathbf{g}(\mathbf{x}))$ because $\mathbf{x}$ obviously satisfies the constraints if each c_j^* is replaced by $g_j(\mathbf{x})$.

But then

$$f^*(\mathbf{g}(\mathbf{x})) \;\le\; f^*(\mathbf{g}(\mathbf{x}) + \underbrace{\mathbf{c}^* - \mathbf{g}(\mathbf{x}^*)}_{\ge 0})$$

since $\mathbf{g}(\mathbf{x}^*) \le \mathbf{c}^*$ and f^* is non-decreasing.

Hence

$$\phi(\mathbf{x}) \;:=\; f(\mathbf{x}) - f^*(\underbrace{\mathbf{g}(\mathbf{x}) + \mathbf{c}^* - \mathbf{g}(\mathbf{x}^*)}_{=:\mathbf{u}(\mathbf{x})}) \;\le\; 0$$

for all $\mathbf{x}$ and since $\phi(\mathbf{x}^*) = 0$, $\phi(\mathbf{x})$ has a maximum in $\mathbf{x} = \mathbf{x}^*$, so

$$\begin{aligned} 0 \;=\; \frac{\partial \phi}{\partial x_i}(\mathbf{x}^*) \;&=\; \frac{\partial f}{\partial x_i}(\mathbf{x}^*) - \sum_{j=1}^m \frac{\partial f^*}{\partial u_j}(\mathbf{u}(\mathbf{x}^*))\,\frac{\partial u_j}{\partial x_i}(\mathbf{x}^*) \\ &=\; \frac{\partial f}{\partial x_i}(\mathbf{x}^*) - \sum_{j=1}^m \frac{\partial f^*}{\partial u_j}(\mathbf{c}^*)\,\frac{\partial g_j}{\partial x_i}(\mathbf{x}^*) \end{aligned}$$

Since f^* is non-decreasing, we have

$$\lambda_j^* \;:=\; \frac{\partial f^*}{\partial c_j}(\mathbf{c}) \;\ge\; 0$$

and we should (but will not) prove that if $g_j(\mathbf{x}^*) < c_j^*$ then $\lambda_j^* = 0$. □

How should we solve a maximization problem by Karush-Kuhn-Tucker? Let's have a look at two examples.

Always: $\lambda_j \geq 0$ and if $g_j(\mathbf{x}) < c_j$ then $\lambda_j = 0$. Respect the direction of the implication!

Not true: If $\lambda_j = 0$ then $g_j(\mathbf{x}) < c_j$.

Example 3.2

$$\begin{array}{rl} \textit{max} & f(x_1, x_2) = x_1^2 + x_2^2 + x_2 - 1 \\ \textit{subsect to} & g(x_1, x_2) = x_1^2 + x_2^2 \leq 1 \end{array}$$

1. *We have one constraint and need one Lagrange-multiplicator $\lambda = \lambda_1$. The Lagrange-function is:*

$$L(x_1, x_2) \;=\; x_1^2 + x_2^2 + x_2 - 1 - \lambda\,(x_1^2 + x_2^2 - 1)$$

2. *Write down the Karush-Kuhn-Tucker-conditions*

(I)	$L_{x_1}(x_1, x_2) \;=\; 2x_1 - 2\lambda x_1 \;=\; 2x_1(1-\lambda)$	$= 0$
(II)	$L_{x_2}(x_1, x_2) \;=\; 2x_2 + 1 - 2\lambda x_2$	$= 0$
(III)	$\lambda \geq 0$ *and* $\lambda(x_1^2 + x_2^2 - 1) \;=\; 0$	

3. *Find all points (x_1, x_2, λ) which satisfy all Karush-Kuhn-Tucker-conditions and pay attention that for all these points $x_1^2 + x_2^2 \leq 1$ (constraint).*

   Systematic way

   From equation (I) we see, that $\lambda = 1$ **or** *$x_1 = 0$. The case $\lambda = 1$ with equation (II) gives a contradiction.* **Hence:** $x_1 = 0$**.**

   All constraints could be binding (=) or not binding (<) and there are 2 possibilities, shortened by = and <.

=	$x_1^2 + x_2^2 = 1$	$\Rightarrow$	$\lambda \geq 0$	*with (III), first part*
<	$x_1^2 + x_2^2 < 1$	$\Rightarrow$	$\lambda = 0$	*with (III), second part*

   (a) *Case = (or $x_1^2 + x_2^2 = 1$)*
   Then with $x_1 = 0$ we get $x_2 = \pm 1$. By (II) we can compute the associated λ and get the two candidats for maximization: $(0, 1, 3/2)$ *and* $(0, -1, 1/2)$

   (b) *Case < (or $x_1^2 + x_2^2 < 1$)*
   With $\lambda = 0$ and $x_1 = 0$ we get by (II) that $x_2 = -1/2$. We have found a third candidat for maximization: $(0, -1/2, 0)$.

   With

$$f(0,1) \;=\; 1, \qquad f(0,-1) \;=\; -1 \qquad \textit{and} \quad f(0,-1/2) \;=\; -5/4$$

   we see that $(0, 1)$ (with $\lambda = 3/2$) is the solution of the maximization problem.

Example 3.3

$$\max \quad y = f(m,x) \ = \ m + \ln x$$

$$\textit{subject to} \quad \begin{cases} g_1(m,x) = m + x \le 5 \\ g_2(m,x) = -m \le 0 \\ g_3(m,x) = -x \le 0 \end{cases}$$

1. *We have three constraints and need three Lagrange-multiplicator* $\lambda_1, \lambda_2, \lambda_3$. *The Lagrange-function is:*

$$\begin{aligned} L(x_1,x_2) &= m + \ln x - \lambda_1\,(m + x - 5) - \lambda_2\,(-m) - \lambda_3\,(-x) \\ &= m + \ln x - \lambda_1\,(m + x - 5) + \lambda_2 m + \lambda_3 x \end{aligned}$$

2. *Write down the Karush-Kuhn-Tucker-conditions*

(I)	$L_{x_1}(x_1,x_2) \ = \ 1 - \lambda_1 + \lambda_2$	$= 0$
(II)	$L_{x_2}(x_1,x_2) \ = \ \frac{1}{x} - \lambda_1 + \lambda_3$	$= 0$
(III)	$\lambda_1 \ge 0$ *and* $\lambda_1(m + x - 5) \ = \ 0$	
(IV)	$\lambda_2 \ge 0$ *and* $\lambda_2(-m) \ = \ 0$	
(V)	$\lambda_3 \ge 0$ *and* $\lambda_3(-x) \ = \ 0$	

3. *Find all points* $(x_1, x_2, \lambda_1, \lambda_2, \lambda_3)$ *which satisfy all Karush-Kuhn-Tucker-conditions and all constraints.*

Systematic way

All constraints could be binding (=) or not binding (<) and there are $2 \cdot 2 \cdot 2 = 8$ *possibilities. Of course, some of these combinations are obviously impossible.*

$(=,=,=)$	$m + x = 5$	$-m = 0$	$-x = 0$	$\Rightarrow$	$\lambda_1 \ge 0$	$\lambda_2 \ge 0$	$\lambda_3 \ge 0$	*no solution*
$(<,=,=)$	$m + x < 5$	$-m = 0$	$-x = 0$	$\Rightarrow$	$\lambda_1 = 0$	$\lambda_2 \ge 0$	$\lambda_3 \ge 0$	*no solution*
$(=,<,=)$	$m + x = 5$	$-m < 0$	$-x = 0$	$\Rightarrow$	$\lambda_1 \ge 0$	$\lambda_2 = 0$	$\lambda_3 \ge 0$	*no solution*
$(=,=,<)$	$m + x = 5$	$-m = 0$	$-x < 0$	$\Rightarrow$	$\lambda_1 \ge 0$	$\lambda_2 \ge 0$	$\lambda_3 = 0$	*no solution*
$(<,<,=)$	$m + x < 5$	$-m < 0$	$-x = 0$	$\Rightarrow$	$\lambda_1 = 0$	$\lambda_2 = 0$	$\lambda_3 \ge 0$	*no solution*
$(<,=,<)$	$m + x =<$	$-m = 0$	$-x < 0$	$\Rightarrow$	$\lambda_1 = 0$	$\lambda_2 \ge 0$	$\lambda_3 = 0$	*no solution*
$(=,<,<)$	$m + x = 5$	$-m < 0$	$-x < 0$	$\Rightarrow$	$\lambda_1 \ge 0$	$\lambda_2 = 0$	$\lambda_3 = 0$	$(4,1,1,0,0)$
$(<,<,<)$	$m + x < 5$	$-m < 0$	$-x < 0$	$\Rightarrow$	$\lambda_1 = 0$	$\lambda_2 = 0$	$\lambda_3 = 0$	*no solution*

Confirm all these results!

Elegant way

If $m + x < 5$ *then* $\lambda_1 = 0$ *by* (III) *and* $1 + \lambda_2 = 0$ *or* $\lambda_2 = -1 < 0$ *by* (I) *which contradicts* (IV). *Hence* $m+x = 5$ *and we have to check only 4 possibilities* $(=,*,*)$.

Because $\ln(x)$ *is not defined in* $x = 0$ *(and by equation* (II)*) we see that* $-x < 0$ *(resp.* $x > 0$*). Hence we have to check the two possibilities* $(=,*,<)$.

- $(=,=,<)$ *means* $m + x = 5$, $m = 0$ *and* $x > 0$ *(and* $\lambda_3 = 0$ *by* (V)*). Then* $m = 5$ *and* $\lambda_1 = 1/5$ *by* (II), $\lambda_2 = -4/5$ *by* (I). *This contradicts* (IV).
- $(=,<,<)$ *means* $m + x = 5$, $m > 0$ *and* $x > 0$ *(and* $\lambda_2 = \lambda_3 = 0$ *by* (IV) *and* (V)*). Then* $\lambda_1 = 1$ *by* (I), $x = 1$ *and* $m = 5 - 1 = 4$. *We get the unique solution* $(4,1,1,0,0)$.