Faculty of Business and Economics
University of Basel
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Course:	**18545-01 Advanced mathematics for economics** MOCK EXAM
Examinator:	**Dr. Thomas Zehrt**
Begin of examination:	
Duration:	**90 minutes**

Important:

- Allowed electronic means: simple pocket calculator (einfacher Taschenrechner, according to Merkblatt Hilfsmittel)
- Allowed non-electronic means: open-book
- The examination consists of 4 questions. Answer ALL the questions.
- Sketches and diagrams are NOT necessarily drawn to scale.
- It is in your own interest to write legibly and to present the work neatly.
- Solve all questions on separate sheets.
- **Cheating and other misconduct, whether attempted or successful, will be penalised by getting the mark 1.0 (BA-Ordnung 17, MA-Ordnung 15).**

Question 1

Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & \sqrt{2} \\ 0 & \sqrt{2} & 1 \end{pmatrix}$.

1. Compute the rank of $\mathbf{A}$. [2]
2. Determine $\det(\mathbf{A})$. Computation or explanation! [2]
3. Determine all eigenvalues of $\mathbf{A}$. [6]
4. Determine the eigenvectors of $\mathbf{A}$ associated to the eigenvalues. [6]
5. Is $\mathbf{A}$ diagonalizable? Explanation! [2]
6. Is $\mathbf{A}$ invertible? Explanation! [2]

Question 2

Let $f(\mathbf{x}) = f(x_1, x_2, x_3) = x_1^2 - 4x_1x_2 + 9x_2^2 + 2x_3^2$.

1. Determine the gradient ∇f and the Hesse matrix $\nabla^2 f$ of f. [5]
2. Determine all points $\mathbf{x} \in \mathbb{R}^3$ with $\nabla f(\mathbf{x}) = \mathbf{0}$. [5]
3. Investigate the definiteness of the Hesse matrix $\nabla^2 f$. [5]
4. Is f concave, convex, quasi-concave or quasi-convex? [5]

Question 3

Consider the problem

$$\max \quad : f(x,y) = xy + x^2 + y$$

$$\text{subject to} \quad : \begin{cases} g_1(x,y) = x^2 + y \leq 2 \\ g_2(x,y) = -y \leq 1 \\ g_3(x,y) = -x \leq 1 \end{cases}$$

1. Sketch the admissible set (given by the constrained functions) of the maximization problem. [3]
2. Write down the Kuhn-Tucker conditions of the problem. [5]
3. Solve comprehensibly and step by step the Kuhn-Tucker conditions only in the (two) cases
   - $x^2 + y < 2$, $-y = 1$ and $-x = 1$; [6]
   - $x^2 + y = 2$, $-y < 1$ and $-x < 1$. [6]

Question 4

1. Prove or disprove: Let $z = f(x, y) = x^a y^b$ be a (differentiable) homogeneous function of degree 1. Then $\det(\nabla^2 f) = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = 0.$ 5

2. Prove or disprove: Let $z = f(x, y) = x^a y^b$ be a (differentiable) homogeneous function of degree 2. Then $\det(\nabla^2 f) = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = 0.$ 5

3. Use the identity $\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$ for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ to prove:

   (a) $\det(\lambda \mathbf{A^{-1}}) = \dfrac{\lambda^{\mathbf{n}}}{\det(\mathbf{A})}$. 3

   (b) $\mathbf{A}$ and $\mathbf{P^{-1}AP}$ have the same eigenvalues. 3

   (c) If λ is an eigenvalue with associated eigenvector $\mathbf{x}$ for the matrix A, then λ is an eigenvalue with associated eigenvector $\mathbf{y} = \mathbf{P^{-1}x}$ for the matrix $\mathbf{P^{-1}AP}$. 4