Differential equations

Keywords: differential processes, differential equations, systems of differential equations

Contents

1 Introduction

Definition 1.1 *A differential equation of order n takes the form*

$$x^{(n)} = f\left(t, x, x^{(1)}, x^{(2)}, \ldots, x^{(n-1)}\right) \qquad (*)$$

where f is a given function in $n+1$ variables and $x = x(t)$ is the unknown function and $x^{(j)} = x^{(j)}(t) = \frac{d^j}{dt^j}x(t)$ is the j-th derivative of x.

Example 1.1
First order differential equations:

- $\dot{x} = t + x,$
- $\dot{x} = \sin(tx),$
- $\dot{x} + 2x = t^2$

Second order differential equations:

- $\ddot{x} = t + x + \dot{x}$,
- $\ddot{x} = \sin(tx)$

Definition 1.2 *A solution of the differential equation $(*)$ on an interval I is an n-times differentiable function satisfying the equation $(*)$.*

Example 1.2 *Solve $\dot{x} = c$.*
Direct integration implies

$$x(t) = \int c dt = ct + A$$

for some constant A. The solutions represent (geometrically) a collection of straight lines.

Example 1.3 *Solve $\ddot{x} = c$.*
Direct integration implies

$$\dot{x}(t) = \int c dt = ct + A$$

for some constant A and

$$x(t) = \int (ct + A) dt = \frac{1}{2}ct^2 + At + B$$

for some constant B. The solutions represent (geometrically) a collection of parabolas.

Definition 1.3 *Let* $x^{(n)} = f\left(t, x, x^{(1)}, x^{(2)}, \ldots, x^{(n-1)}\right)$ *be a differential equation. In an* *initial value problem, there are specified values* $x_0^{(0)}, x_0^{(1)}, \ldots, x_0^{(n-1)} \in \mathbb{R}$ *such that*

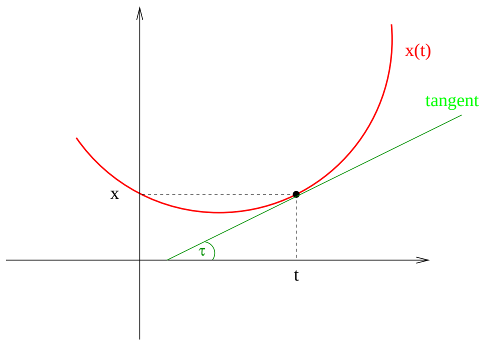
$$x^{(j)}(t_0) = x_0^{(j)} \qquad \text{for all } j = 0, 1, \ldots, n-1.$$

Example 1.4 *Find the solution of* $\dot{x} = 2$ *that passes through the point* $(t, x) = (0, 1)$ *(or* $x(0) = 1$*). We know that the general solution of the differential equation is* $x(t) = 2t + A$*. We must have* $1 = x(0) = A$ *and the solution of the initial value problem is* $x(t) = 2t + 1$*.*

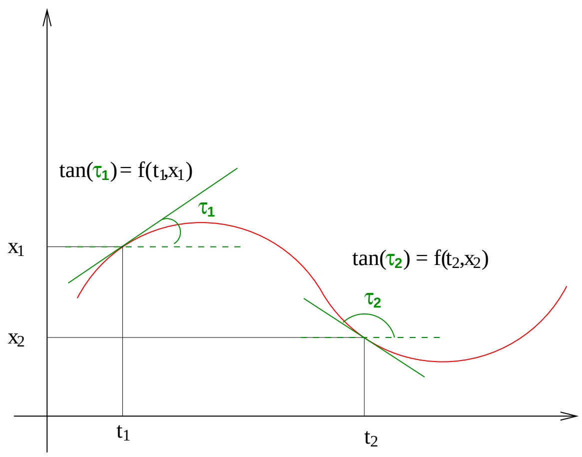
2 First-order differential equations

2.1 Geometric interpretation of $\dot{x}(t) = f(t, x(t))$

$$\underbrace{\dot{x}(t)}_{\substack{\text{slope of the} \\ \text{solution } x(t) \text{ in } (t,x)}} \quad = \quad \underbrace{f(t,x)}_{\substack{\text{value of } f \\ \text{in}(t,x)}}$$

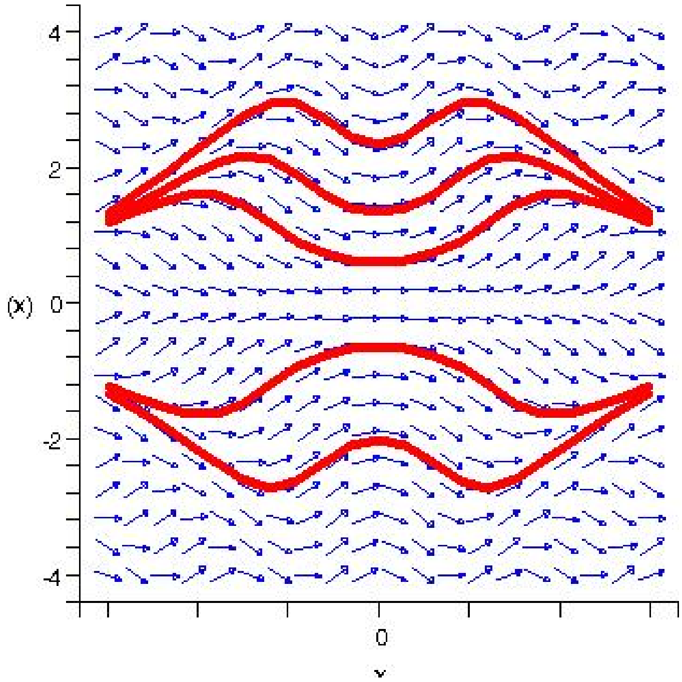


In the (t, x)-plane it means: If the graph of a solution $x(t)$ meets the point (t, x), then it must have the slope $f(t, x)$ there.



- If $x(t)$ is a solution of $\dot{x}(t) = f(t, x(t))$, then the slope of the tangent to the graph at the point (t, x) is $f(t, x)$.
- We can draw small straight-line segments with slopes $f(t, x)$ through several points in the tx-plane.
- This gives us a so-called direction diagram or slope field for the differential equation.
- If the graph of a solution passes through one of these points, it has the corresponding line segment as its tangent.

Example 2.1 *slope field of* $\dot{x}(t) = \sin(tx)$ *with some solutions*



2.2 Separable equations

Definition 2.1 *A differential equation $\dot{x} = f(t, x)$ is called separable, if the function f can be written as a product of two functions, one depending only on t and the other only on x.*

$$\boxed{\dot{x} = g(t) \cdot h(x)}$$

A particular solution of a separable equation arises if $h(x)$ has a zero at $x = a$, so that $h(a) = 0$. In this case $x(t) = a$ will be a solution of the equation.

Example 2.2 *The differential equation $\dot{x} = (x + 1)(x - 3)$ has the two particular solutions $x(t) = -1$ and $x(t) = 3$.*

Method for solving separable differential equations $\dot{x} = \dfrac{dx}{dt} = g(t) \cdot h(x)$

1. Every zero $x = a$ of $h(x)$ gives the constant solution $x(t) = a$.
2. Separate the variables

   „$h(x)$ and dx to the left" und „$g(t)$ and dt to the right"

   $$\frac{1}{h(x)}\, dx = g(t)\, dt$$

3. Integrate each side:

   $$H(x) := \int \frac{1}{h(x)}\, dx \qquad G(t) := \int g(t)\, dt$$

   The solution (in implicit form) is $H(x) - G(t) = c$.

4. Solve for x, if possible.

Example 2.3

$$\dot{x} = \frac{dx}{dt} = x^2 = h(x)$$

1. *$h(0) = 0$ and $x(t) = 0$ is a constant solution.*
2. *Separate the variables:* $\dfrac{1}{x^2}\, dx = dt$
3. $H(x) := \displaystyle\int \frac{1}{x^2}\, dx = -x^{-1} \qquad G(t) := \int dt = t$

   and the solution (implicit) is $-x^{-1} - t = c$.

4. *Solve for x:* $x(t) = -\dfrac{1}{t + c}$.

2.3 First-order linear equations

Definition 2.2 *A first-order linear differential equation can be written in the form*

$$\boxed{\dot{x} + a(t)\, x \;=\; b(t)}$$

where a and b denote continuous functions of t in a certain interval. The equation is called homogeneous, if $b(t) = 0$.

Theorem 2.1 *Let a be a continuous function. Then the general solution of $\dot{x} + a(t)\, x \;=\; 0$ is given by*

$$x_h(t) \;=\; C \cdot e^{-A(t)}$$

with $A(t) = \int a(t)dt$.

Proof: $\dot{x} + a(t)\, x \;=\; 0$ is separable and we get

$$\begin{aligned}
& \dot{x} + a(t)\, x \;=\; 0 \\
\Rightarrow\quad & \frac{dx}{x} = -a(t)dt \\
\Rightarrow\quad & \ln(|x|) = -\int a(t)dt + C_1 \;=\; -A(t) + C_1 \\
\Rightarrow\quad & x_h(t) = \underbrace{e^{C_1}}_{=:C} \cdot e^{-A(t)}
\end{aligned}$$

□

We will now solve the nonhomogeneous equation $\dot{x} + a(t)\, x \;=\; b(t)$. We can see that the difference (function) of two arbitrary solutions $x_1 = x_1(t)$ and $x_2 = x_2(t)$ of this equation is a solution of the **homogeneous** equation:

$$\begin{aligned}
\frac{d}{dt}(x_1 - x_2) + a(t)(x_1 - x_2) \;&=\; \dot{x}_1 + a(t)\, x_1 - \dot{x}_2 - a(t)\, x_2 \\
&=\; b(t) - b(t) \;=\; 0
\end{aligned}$$

Theorem 2.2 *The general solution of $\dot{x} + a(t)\, x \;=\; b(t)$ can be written in the form*

$$x(t) \;=\; x^p(t) + x^h(t) \;=\; x^p(t) + ce^{-A(t)}$$

with any (single) solution $x^p(t)$ of the nonhomogeneous equation and the general solution $x^h(t)$ of the homogeneous equation.

How can we find a particular solution $x^p(t)$ for $\dot{x} + a(t)\, x \;=\; b(t)$?

One trick is to guess the structure of the solution $x^p \;=\; c(t)x^h$ (variation of the constant c). Then we try to determine the suitable function $c(t)$.

$$
\begin{aligned}
b(t) \;=\; \dot{x}^p + a(t)\, x^p \;&=\; \frac{d}{dt}(c(t)\cdot x^h) + a(t)\, c(t)\cdot x^h \\
&=\; \dot{c}(t)\cdot x^h + c(t)\cdot \dot{x}^h + a(t)\, c(t)\cdot x^h \\
&=\; \dot{c}(t)\cdot x^h + c(t)\underbrace{(\dot{x}^h + a(t)\, x^h)}_{=0} \\
&=\; \dot{c}(t)\cdot x^h
\end{aligned}
$$

It follows that

$$
\dot{c}(t) \;=\; \frac{b(t)}{x^h} \quad \text{hence} \quad c(t) \;=\; \int \frac{1}{x^h(t)}\, b(t)\, dt + C,
$$

with $x^h(t) = e^{-A(t)}$ and so

Theorem 2.3 *The general solution of* $\dot{x} + a(t)\, x \;=\; b(t)$ *is given by*

$$
\begin{aligned}
x(t) \;&=\; e^{-A(t)} \left(\int e^{A(t)}\, b(t)\, dt + C \right) \\
&=\; e^{-A(t)} \int e^{A(t)}\, b(t)\, dt + Ce^{-A(t)}
\end{aligned}
$$

with $A(t) = \int a(t)dt$.

Example 2.4 *Find the general solution of* $\dot{x} + 2tx \;=\; 4t$.

- $a(t) = 2t$ *and* $b(t) = 4t$
- $A(t) \;=\; \int a(t)dt \;=\; t^2 + C(=0)$
- $$
x(t) \;=\; e^{-t^2} \left(\int e^{t^2}\, 4t\, dt + C \right) \;=\; Ce^{-t^2} + 2.
$$

3 Second-order equations

3.1 Linear differential equations

Definition 3.1 *The general second-order linear differential equation can be written as*

$$\boxed{\ddot{x} + a(t)\,\dot{x} + b(t)\,x = \; c(t)}$$

where a, b and c are all continuous functions of t in a certain interval. The equation is called homogeneous, if $c(t) = 0$.

Example 3.1 $\ddot{x} - \;\dot{x} + \;x \;=\; 0$ *and* $\ddot{x} + t^2\,\dot{x} + t\,x \;=\; 2t + 1$

The homogeneous equation $\ddot{x} + a(t)\,\dot{x} + b(t)\,x \;=\; 0$ $\quad(*)$

- If $x_1 = x_1(t)$ and $x_2 = x_2(t)$ both satisfy equation $(*)$, then so does $x = Ax_1 + Bx_2$ for all choices of constants A and B.
- Suppose we have somehow managed to find two solutions x_1 and x_2. Does the general solution take the form $x = Ax_1 + Bx_2$? **No!** We must requiere $x_1 \neq k \cdot x_2$!

Theorem 3.1 *The* **general solution** *of*

$$\boxed{\ddot{x} + a(t)\,\dot{x} + b(t)\,x \;=\; 0 \quad \textit{is} \quad x = Ax_1 + Bx_2 \quad (A, B \in \mathbb{R})}$$

where $x_1 = x_1(t)$ and $x_2 = x_2(t)$ are any two solutions that are not proportional.

The nonhomogeneous equation $\ddot{x} + a(t)\,\dot{x} + b(t)\,x \;=\; c(t)$

The difference (function) of two arbitrary solutions of this nonhomogeneous equation is a solution of the **homogeneous** equation!

Theorem 3.2 *The* **general solution** *of*

$$\boxed{\ddot{x} + a(t)\,\dot{x} + b(t)\,x \;=\; c(t) \quad \textit{is} \quad x \;= Ax_1 + Bx_2 + x^p}$$

where $Ax_1 + Bx_2$ is the general solution of the associated homogeneous equation and x^p is any particular solution of the nonhomogeneous equation.

3.2 Linear differential equations with constant coefficients

Consider the equation

$$\ddot{x} + a\,\dot{x} + b\,x = \ c \qquad a, b, c \in \mathbb{R}.$$

The general solution can be written as $x \ = Ax_1 + Bx_2 + x^p$ with $Ax_1 + Bx_2$ the general solution of the homogeneous equation and x^p a particular solution of the nonhomogeneous equation.

For the equation

$$\ddot{x} + a\,\dot{x} + b\,x = \ 0 \qquad a, b \in \mathbb{R}$$

it seems a good idea to try solutions x with the property that x, $\dot{x}$ and $\ddot{x}$ are all constant multiples of each other. The functions $x = e^{rt}$ have this property and we can try adjust the constant r in order that $x = e^{rt}$satisfies the equation.

We have:

If $x = e^{rt}$ is a solution of $\ddot{x} + a\,\dot{x} + b\,x = \ 0$

$\longrightarrow \quad r^2\,e^{rt} + ar\,e^{rt} + b\,e^{rt} = \ 0$

$\longrightarrow \quad r^2 + ar + b = \ 0$

Definition 3.2 $r^2 + ar + b = \ 0$ *is called the characteristic equation of* $\ddot{x} + a\,\dot{x} + b\,x = \ 0$.

Theorem 3.3 *The general solution of*

$$\ddot{x} + a\,\dot{x} + b\,x = \ 0 \qquad a, b \in \mathbb{R}$$

depends on the the characteristic equation as follows:

1. *If* $\frac{a^2}{4} - b > 0$ *(two distinct real roots) then*

$$x = Ae^{r_1 t} + Be^{r_2 t} \quad \textit{with} \quad r_{1,2} = -\tfrac{a}{2} \pm \sqrt{\tfrac{a^2}{4} - b}$$

2. *If* $\frac{a^2}{4} - b = 0$ *(a double real root) then*

$$x = (A + Bt)e^{rt} \quad \textit{with} \quad r = -\tfrac{a}{2}$$

3. *If* $\frac{a^2}{4} - b < 0$ *(no real roots) then*

$$x = e^{\alpha t}(A\cos\beta t + B\sin\beta t) \quad \textit{with} \quad \alpha = -\tfrac{a}{2},\ \beta = \sqrt{b - \tfrac{a^2}{4}}$$

Example 3.2 *Find the general solutions of the following equations:*

- $\ddot{x} - 3x = 0$

  $x = Ae^{-\sqrt{3}t} + Be^{\sqrt{3}t}$

- $\ddot{x} - 4\dot{x} + 4x = 0$

  $x = (A + Bt)e^{2t}$

- $\ddot{x} - 6\dot{x} + 13x = 0$

  $x = e^{3t}(A\cos 2t + B\sin 2t)$

Theorem 3.4 *A particular solution of*

$$\ddot{x} + a\,\dot{x} + b\,x \;=\; c$$

is

$$x^p = \begin{cases} \dfrac{c}{b} & \text{if } b \neq 0 \\[2ex] \dfrac{c}{a}\,t & \text{if } b = 0 \text{ and } a \neq 0 \\[2ex] \dfrac{c}{2}\,t^2 & \text{if } b = 0 \text{ and } a = 0 \end{cases} .$$

4 Systems of differential equations

Definition 4.1 *Let f, g be functions in 3 variables. A* <u>*normal system*</u> *of 2 first-order differential equations in 2 variables takes the form*

$$\begin{aligned} \frac{dx(t)}{dt} &= \dot{x} &= f(x, y, t) \\ \frac{dy(t)}{dt} &= \dot{y} &= g(x, y, t) \end{aligned}$$

A solution is a pair of differentiable functions $(x(t), y(t))$, defined on some interval I, that satisfies both equations.

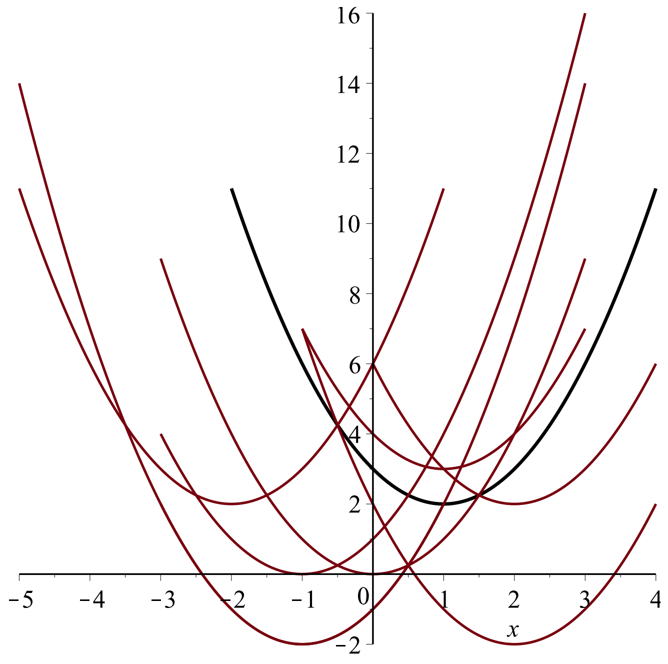
Theorem 4.1 *If f, g, f_x, f_y, g_x, g_y are continuous then we have the follwing fact:*

If $t_0 \in I$ and $x_0, y_0 \in \mathbb{R}$ there is **one and only one** *pair of functions $(x(t), y(t))$ that satisfies the two equations and $x(t_0) = x_0$ and $y(t_0) = y_0$.*

Example 4.1

$$\begin{aligned} \dot{x} &= 1 \\ \dot{y} &= 2t \end{aligned}$$

Integrate each equation directly: $x(t) = t + C_1$ and $y(t) = t^2 + C_2$ or $y = (x - C_1)^2 + C_2$. With the initial condition $x_0 = 1$ and $y_0 = 2$ we get the solution $x(t) = t + 1$ and $y(t) = t^2 + 2$ or $y = (x - 1)^2 + 2$.



How can we find a general solution of the system

$$\begin{aligned} \dot{x} &= f(t,x,y) \\ \dot{y} &= g(t,x,y) \end{aligned}$$

We can **not** expect exact methods that work in complete generality! One important method: Reduce the system to a (single) second-order differential equation in the following way.

- Use the first equation to express y as a function $y = h(t, x, \dot{x})$,
- Differentiate this equation with respect to t and
- Substitute the expressions for y and $\dot{y}$ in the second equation.

Example 4.2 *Find the general solution of the system*

$$\begin{aligned} (I)\ \dot{x} &= 2x + e^t y - e^t \\ (II)\ \dot{y} &= 4e^{-t}x + y \end{aligned}$$

Solution:

- $(I) \leftrightarrow y = e^{-t}\dot{x} - e^{-t}2x + 1 \rightarrow \dot{y} = -e^{-t}\dot{x} + e^{-1}\ddot{x} + e^{-t}2x - e^{-t}2\dot{x}$
- *in* (II) $\dot{y} = 4e^{-t}x + y \leftrightarrow 0 = \ddot{x} - 4\dot{x} - e^t$
- *General solution:* $x(t) = C_1 + C_2 e^{4t} + \frac{1}{3}e^t$
- $y(t) = e^{-t}\dot{x} - e^{-t}2x + 1 = \dots$