# Static optimization

**Keywords:** local and global extremal points, unconstrained optimization, constrained optimization, Lagrange function, Lagrange multipliers, Lagrange method, Kuhn-Tucker method

## Contents

# 1 Overview about (static) optimization problems

In a general static optimization problem there is

- a real-valued function

$$f(\mathbf{x}) \;=\; f(x_1, \ldots, x_n)$$

  in $n$ variables, the so-called objective function, whose value is to be optimized (maximized or minimized) and

- a set $D \subset \mathbb{R}^n$, the so-called admissible set.

Then the problem is to find (global) maximum or minimum points $\mathbf{x}^* \in D$ of $f$:

$$\max(\min)\ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D.$$

**From now on we will always assume that $f$ is at least 2-times continuously partial differentiable.**

Because $\max\ f(\mathbf{x}) \;=\; \min\ -f(\mathbf{x})$ subject to $\mathbf{x} \in D$ we could focus our attention (without loss of generality) on minimizing problems.

Depending on the set $D$ and the function $f$ several different types of optimization problems can arise. At the first level we will distinguish between so-called

1. unconstrained optimization problems:

   $D$ contains no boundary points of $D$. This means that the set $D$ is an open subset of $\mathbb{R}^n$ and a solution of the optimization problem (if it exists) is an interrior point of $D$.

   **Example 1.1** *Solve the following problems or explain why there are no solutions:*

   $\min x^2$ *subject to* $x \in D = (-1, 1)$

   $\min -x^2$ *subject to* $x \in D = (-1, 1)$

   $\min x^2$ *subject to* $x \in D = \mathbb{R}$

   $\min 1/x$ *subject to* $x \in D = (0, 1)$

   $\min -1/x$ *subject to* $x \in D = (0, 1)$

   $\min x^2 - x^4$ *subject to* $x \in D = (-2, 2)$

   $\min x^2 - x^4$ *subject to* $x \in D = (-1, 1)$

   $\min x^2 - x^4$ *subject to* $x \in D = (-0.1, 0.1)$

   $\min \sin(1/x)/x$ *subject to* $x \in D = (0, 1)$

2. constrained optimization problems:

   $D$ contains some boundary points of $D$. A solution of the optimization problem may be an interior point or a point on the boundary of $D$.

# 2 Unconstrained optimization problems

## 2.1 Local minimizer

Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. Let $D$ be some **open** subset of $\mathbb{R}^n$ and $\mathbf{x}^* \in D$ a local minimizer of $f$ over $D$. This means that there exists an $\epsilon > 0$ such that for all $\mathbf{x} \in D$ satisfying $|\mathbf{x} - \mathbf{x}^*| < \epsilon$ we have $f(\mathbf{x}^*) \leq f(\mathbf{x})$.

The term ,,unconstrained" usually refers to the situation where all points $\mathbf{x}$ sufficiently near $\mathbf{x}^*$ are in $D$. This is automatically true if $D$ is an open set.

We already know:

**Theorem 2.1 (First- and second order necessary conditions for optimality)**
*Suppose that $\nabla^2 f$ is continuous in an open neighbourhood $U$ of $\mathbf{x}^*$ then*

$$\boxed{\mathbf{x}^* \text{ is a local minimizer of } f \implies \nabla f(\mathbf{x}^*) = \mathbf{0} \text{ and } \nabla^2 f(\mathbf{x}^*) \text{ is pos.semidef.}}$$

Note that these necessary conditions are not sufficient.

**Theorem 2.2 (First- and second order sufficient conditions for optimality)**
*Suppose that $\nabla^2 f$ is continuous in an open neighbourhood $U$ of $\mathbf{x}^*$ then*

$$\boxed{\nabla f(\mathbf{x}^*) = \mathbf{0} \text{ and } \nabla^2 f(\mathbf{x}^*) \text{ is pos.def.} \implies \mathbf{x}^* \text{ is a (strict) local minimizer of } f}$$

**Proof:**

Because $\nabla^2 f$ is continuous and positive definite at $\mathbf{x}^*$, we can choose an open ball $B = \{\mathbf{x} \mid ||\mathbf{x} - \mathbf{x}^*|| < \epsilon\} \subset D$ where $\nabla^2 f$ remains positive definite. Taking any nonzero vector $\mathbf{v}$ with $||\mathbf{v}|| < \epsilon$, we have $\mathbf{x}^* + \mathbf{v} \in B$ and by Taylor's theorem:

$$\begin{aligned} f(\mathbf{x}^* + \mathbf{v}) &= f(\mathbf{x}^*) + \mathbf{v}^T \nabla f(\mathbf{x}^*) + \frac{1}{2}\, \mathbf{v}^T \nabla^2 f(\mathbf{z}) \mathbf{v} \\ &= f(\mathbf{x}^*) + \frac{1}{2}\, \mathbf{v}^T \nabla^2 f(\mathbf{z}) \mathbf{v} \end{aligned}$$

for some $\mathbf{z} = \mathbf{x}^* + t \cdot \mathbf{v}$ with $t \in (0, 1)$.

Since $\mathbf{z} = \mathbf{x}^* + t \cdot \mathbf{v} \in B$, we have $\mathbf{v}^T \nabla^2 f(\mathbf{z}) \mathbf{v} > 0$ and therefore $f(\mathbf{x}^* + \mathbf{v}) > f(\mathbf{x}^*)$. □

## 2.2 Global minimizer

Of course, all local minimizers of a function $f$ are candidates for global minimizing, but obviously, an arbitrary function may not realise a global minimum in an open set $D$. For instance, look at $f(x) = -x^2$ subject to $x \in D = (-1, 1)$.

There are only general results in the case where $f$ is a convex function on $D$. Because we define convexity of the function $f$ by the inequality

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \quad \leq \quad tf(\mathbf{x}) + (1-t)f(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in D$ and all $t \in [0, 1]$, all points $t\mathbf{x} + (1-t)\mathbf{y}$ (points between $\mathbf{x}$ and $\mathbf{y}$) should lie in $D$. Hence $D$ must be a convex set.

**Theorem 2.3** *Let $f$ be a convex (resp. concave) and differentiable function on the convex (and open) set $D$. Then*

$$\boxed{\mathbf{x}^* \text{ is a global minimizer (resp. maximizer) of } f \iff \nabla f(\mathbf{x}^*) \;=\; \mathbf{0}}$$

**Proof** (for convex $f$):

- „$\Longrightarrow$“
  Clear!?
- „$\Longleftarrow$“
  Let $\nabla f(\mathbf{x}^*) \;=\; \mathbf{0}$ and suppose that $\mathbf{x}^*$ is **not** a global minimizer of $f$ on $D$. Then we can find a point $\mathbf{y} \in D$ with $f(\mathbf{y}) < f(\mathbf{x}^*)$.
  Consider the line segment that joins $\mathbf{x}^*$ to $\mathbf{y}$, that is

$$\mathbf{z} \;=\; \mathbf{z}(t) \;=\; t\mathbf{y} + (1-t)\mathbf{x}^* \;=\; \mathbf{x}^* + t(\mathbf{y} - \mathbf{x}^*)$$

  for all $t \in [0, 1]$. Of course, $\mathbf{z} \in D$ because $D$ is a convex set. Hence

$$\begin{aligned}
\nabla f(\mathbf{x}^*)^T\,(\mathbf{y} - \mathbf{x}^*) &= \frac{d}{dt} f(\mathbf{x}^* + t(\mathbf{y} - \mathbf{x}^*))\bigg|_{t=0} \\
&= \lim_{t\to 0+} \frac{f(\mathbf{x}^* + t(\mathbf{y} - \mathbf{x}^*)) - f(\mathbf{x}^*)}{t} \\
&\leq \lim_{t\to 0+} \frac{tf(\mathbf{y}) + (1-t)f(\mathbf{x}^*) - f(\mathbf{x}^*)}{t} \\
&= \lim_{t\to 0+} \frac{t(f(\mathbf{y}) - f(\mathbf{x}^*))}{t} \\
&= f(\mathbf{y}) - f(\mathbf{x}^*) \;<\; 0.
\end{aligned}$$

  Therefore, $\nabla f(\mathbf{x}^*) \;\neq\; \mathbf{0}$! Contradiction.
  Hence, $\mathbf{x}^*$ is a global minimizer of $f$ on $D$.

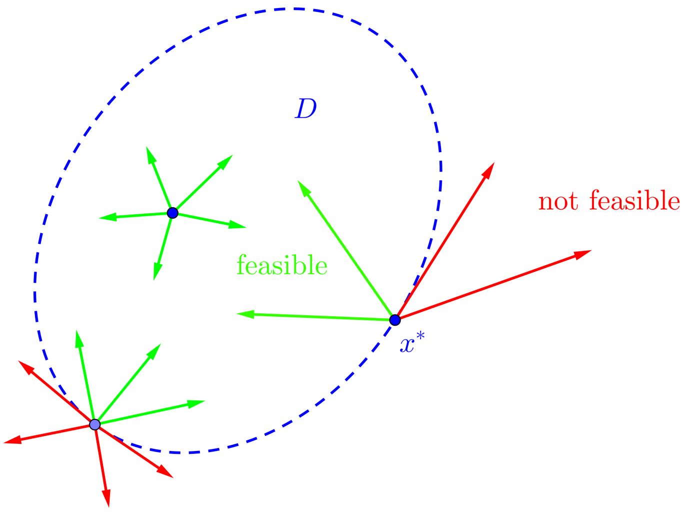
$\square$

# 3 Constrained optimization problems

## 3.1 General remarks

In the previous case we have used the fact that for every direction $\mathbf{v}$ points of the form $\mathbf{x}^* + t\mathbf{v}$ belong to $D$ (for sufficiently small $t$). This is no longer true if $D$ has a boundary and $\mathbf{x}^*$ is a point on this boundary.

**Definition 3.1** *Let $D \subset \mathbb{R}^n$ and $\mathbf{x}^* \in D$. A vector $\mathbf{v} \in \mathbb{R}^n$ is called a feasible direction in $\mathbf{x}^*$ if $\mathbf{x}^* + t\mathbf{v} \in D$ for all $t$ with $0 \le t < t_0$.*




If not all directions $\mathbf{v}$ are feasible in $\mathbf{x}^*$, then the condition $\nabla f(\mathbf{x}^*) = \mathbf{0}$ is no longer necessary for local optimality. But we can prove the following result.

**Theorem 3.1** *If $\mathbf{x}^*$ is a local minimum of the continuous differentiable function $f$ on $D$, then*

$$\partial_{\mathbf{v}} f(\mathbf{x}^*) \;=\; \nabla f(\mathbf{x}^*)^T \mathbf{v} \;\; \ge \;\; 0$$

*for every feasible direction $\mathbf{v}$ and*

$$\mathbf{v}^T \, \nabla^2 f(\mathbf{x}^*) \, \mathbf{v} \;\; \ge \;\; 0$$

*for all feasible directions with $\partial_{\mathbf{v}} f(\mathbf{x}^*) \;=\; 0$.*

There are two cases:

1. $\partial D \not\subset D$

   There are boundary points of $D$ which are not elements of $D$. This case is too difficult and we need a specific method, adapted to the concrete set $D$, to solve the optimization problem. We will **not** follow up on this type of problem.

2. $\partial D \subset D$

   The complete boundary $\partial D$ of $D$ is in $D$; this means that $D$ is closed.

**From now on let $D$ be always closed.**

We recall the following basic existence result for **closed and bounded** sets $D$:

**Theorem 3.2 (Weierstrass-Theorem)** *If $f$ is a continuous function and $D$ is a closed and bounded set then there exists a global minimum of $f$ over $D$.*

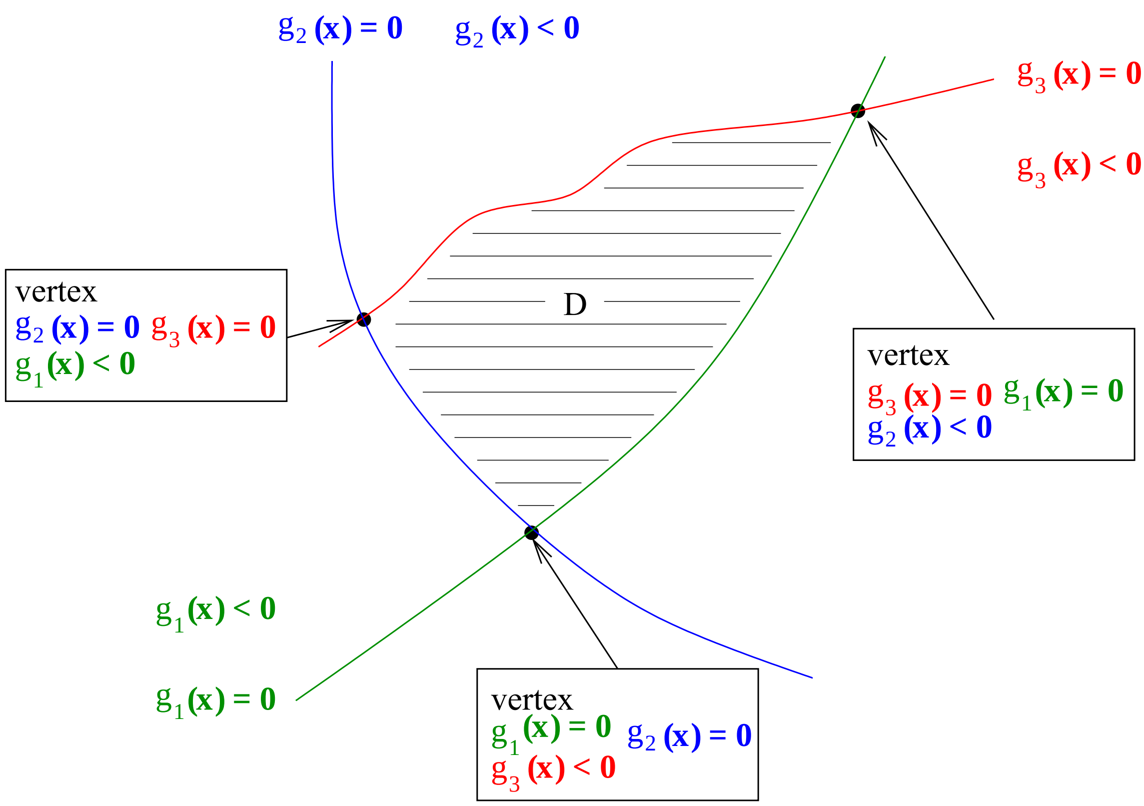
(General) Algorithm for finding a global minimum

1. Find all interior points of $D$ satisfying $\nabla f(\mathbf{x}^*) = \mathbf{0}$ (stationary points).
2. Find all points where $\nabla f$ does not exist (critical points).
3. Find all boundary points satisfying $\partial_{\mathbf{v}} f(\mathbf{x}^*) \geq 0$ for all feasible directions $\mathbf{v}$.
4. Compare all values at all these candidate points and choose one smallest one.

In almost all interesting optimization problems the admissible set $D$ is given by a set of inequalities (or equations):

$$D = \{\mathbf{x} \in \mathbb{R}^n \mid g_1(\mathbf{x}) \leq c_1, g_2(\mathbf{x}) \leq c_2, \ldots, g_m(\mathbf{x}) \leq c_m\} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{g}(\mathbf{x}) \leq \mathbf{c}\}$$

with $\mathbf{g} = (g_1, \ldots, g_m)^T$, $g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{c} = (c_1, \ldots, c_m)^T$.




It is easy to see that one equation of the form $g(\mathbf{x}) = c$ can be expressd by the two inequalities $g(\mathbf{x}) \leq c$ and $-g(\mathbf{x}) \leq -c$. Hence all sets described by a set of equations could be described by a set of inequalites and it would be enough to study sets described by inequalities.

But for practical reasons we will discuss the two cases separately.

**Definition 3.2** *For the optimization problem*

$$\max(\min) \qquad y = f(x_1, x_2, \ldots, x_n) = f(\mathbf{x})$$

$$\textit{subject to} \qquad \begin{cases} g_1(x_1, x_2, \ldots, x_n) = g_1(\mathbf{x}) \le c_1 \\ g_2(x_1, x_2, \ldots, x_n) = g_2(\mathbf{x}) \le c_2 \\ \cdots \\ g_m(x_1, x_2, \ldots, x_n) = g_m(\mathbf{x}) \le c_m \end{cases}$$

*the function (in $n+m$ variables)*

$$L(x_1, \cdots, x_n, \lambda_1, \ldots, \lambda_m) \;=\; f(x_1, x_2, \cdots, x_n) - \sum_{j=1}^{m} \lambda_j \, (g_j(x_1, x_2, \cdots, x_n) - c_j)$$

*shortly*

$$L(\mathbf{x}, \boldsymbol{\lambda}) \;=\; f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j \, (g_j(\mathbf{x}) - c_j) \;=\; f(\mathbf{x}) - \boldsymbol{\lambda}^T(\mathbf{g}(\mathbf{x}) - \mathbf{c})$$

*is called Lagrange function of the optimization problem.*

## 3.2 $D = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{g}(\mathbf{x}) = \mathbf{c}\}$

Given the following optimization problem:

$$\max(\min) \qquad y = f(x_1, x_2, \ldots, x_n) = f(\mathbf{x})$$

$$\text{subject to} \qquad \begin{cases} g_1(x_1, x_2, \ldots, x_n) = g_1(\mathbf{x}) = c_1 \\ g_2(x_1, x_2, \ldots, x_n) = g_2(\mathbf{x}) = c_2 \\ \qquad\cdots \\ g_m(x_1, x_2, \ldots, x_n) = g_m(\mathbf{x}) = c_m \end{cases}$$

**Theorem 3.3** *Suppose that*

- $f, g_1, \ldots, g_m$ *are defined on a set* $S \subset \mathbb{R}^n$
- $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ *is an interior point of* $S$ *that solves the optimization problem*
- $f, g_1, \ldots, g_m$ *are continuously partial differentiable in a ball around* $\mathbf{x}^*$
- *the Jacobi-matrix of the constraint functions*

$$D\mathbf{g}(\mathbf{x}) \;=\; \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}) & \frac{\partial g_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial g_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1}(\mathbf{x}) & \frac{\partial g_m}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial g_m}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

  *has rank* $m$ *in* $\mathbf{x} = \mathbf{x}^*$.

### Necessary condition

*Then there exists unique numbers* $\lambda_1^*, \ldots, \lambda_m^*$ *such that* $(\mathbf{x}^*, \boldsymbol{\lambda}^*) = (x_1^*, x_2^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*)$ *is a stationary point of the Lagrange-function:*

$$\begin{array}{lcl} L_{x_1}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \;=\; 0, \ldots, L_{x_n}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \;=\; 0 & & \\ \qquad\qquad \textit{and} & \qquad \textit{shortly} \qquad & \boxed{\nabla L(\mathbf{x}^*, \boldsymbol{\lambda}^*) \;=\; \mathbf{0}} \\ L_{\lambda_1}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \;=\; 0, \ldots, L_{\lambda_m}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \;=\; 0 & & \end{array}$$

*or expanded*

$$\boxed{\nabla f(\mathbf{x}^*) \;-\; \sum_{j=1}^{m} \lambda_j^* \, \nabla g_j(\mathbf{x}^*) \;=\; \mathbf{0} \qquad (\star)}$$

### Sufficient condition

*If there exist numbers* $\lambda_1^*, \ldots, \lambda_m^*$ *and an admissible* $\mathbf{x}^*$ *which together satisfy the necessary condition, and if the Lagrange function* $L$ *is concave (convex) in* $\mathbf{x}$ *and* $S$ *is convex, then* $\mathbf{x}^*$ *solves the maximization (minimization) problem.*

**Proof:**

**Necessary condition** We get a nice argument for condition ($\star$) by studying the optimal value function

$$f^*(\mathbf{c}) \;=\; \max\{f(\mathbf{x}) \mid \mathbf{g}(\mathbf{x}) = \mathbf{c}\}$$

If $f$ is a profit function and $\mathbf{c} = (c_1, \ldots, c_m)$ denotes a resource vector, then $f^*(\mathbf{c})$ is the maximum profit obtainable given the available resource vector $\mathbf{c}$.

In the following argument we **assume that $f^*(\mathbf{c})$ is differentiable.**

Fix a vector $\mathbf{c}^*$ and let $\mathbf{x}^*$ be the corresponding optimal solution. Then $f(\mathbf{x}^*) = f^*(\mathbf{c}^*)$ and obviously for all $\mathbf{x}$ we have $f(\mathbf{x}) \le f^*(\mathbf{g}(\mathbf{x}))$.

Hence

$$\phi(\mathbf{x}) \;:=\; f(\mathbf{x}) - f^*(\mathbf{g}(\mathbf{x})) \;\le\; 0$$

has a maximum in $\mathbf{x} = \mathbf{x}^*$, so

$$0 \;=\; \frac{\partial \phi}{\partial x_i}(\mathbf{x}^*) \;=\; \frac{\partial f}{\partial x_i}(\mathbf{x}^*) - \sum_{j=1}^{m} \left[\frac{\partial f^*}{\partial c_j}(\mathbf{c})\right]_{\mathbf{c}=\mathbf{g}(\mathbf{x}^*)} \frac{\partial g_j}{\partial x_i}(\mathbf{x}^*)$$

Define

$$\lambda_j^* \;:=\; \frac{\partial f^*}{\partial c_j}(\mathbf{c}) \;\approx\; f^*(\mathbf{c} + \mathbf{e}_j) - f^*(\mathbf{c})$$

and equation ($\star$) follows.

**Sufficient condition** Suppose that $L = L(\mathbf{x})$ is a concave (resp. convex) function in the variable $\mathbf{x}$. The necessary condition means that $\mathbf{x}^*$ is a stationary point of $L$, this means $\nabla_{\mathbf{x}} L(\mathbf{x}^*) = \mathbf{0}$. Then by Theorem 2.3 we know that $\mathbf{x}^*$ is a global maximizer (resp. minimizer) of $L$ and this means that

$$\begin{aligned} L(\mathbf{x}^*) \;&=\; f(\mathbf{x}^*) - \sum_{j=1}^{m} \lambda_j^*(g_j(\mathbf{x}^*) - c_j) \\ &\ge\; f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j^*(g_j(\mathbf{x}) - c_j) \\ &=\; L(\mathbf{x}) \end{aligned}$$

for all $\mathbf{x} \in S$. But for all admissible $\mathbf{x}$ we have $g_j(\mathbf{x}) = c_j$. Hence $f(\mathbf{x}^*) \ge f(\mathbf{x})$ for all admissible $\mathbf{x} \in S$. $\square$

**Example 3.1** *Given the following optimization problem:*

$$\begin{array}{ll} \max & f(x_1, x_2) = x_1^\alpha x_2^\beta \\ \textit{subject to} & g(x_1, x_2) = p_1 x_1 + p_2 x_2 = c \end{array}$$

*Then step by step we get:*

- $L(x_1, x_2, \lambda) \;=\; x_1^\alpha x_2^\beta - \lambda(p_1 x_1 + p_2 x_2 - c)$
- $\nabla L(x_1, x_2, \lambda) \;=\; \nabla f(x_1, x_2) - \lambda \nabla g(x_1, x_2) \;=\; \begin{pmatrix} \alpha x_1^{\alpha-1} x_2^\beta - \lambda p_1 \\ \beta x_1^\alpha x_2^{\beta-1} - \lambda p_2 \\ -(p_1 x_1 + p_2 x_2 - c) \end{pmatrix}$
- $\begin{pmatrix} \alpha x_1^{\alpha-1} x_2^\beta - \lambda p_1 \\ \beta x_1^\alpha x_2^{\beta-1} - \lambda p_2 \\ -(p_1 x_1 + p_2 x_2 - c) \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ *or*

$$\begin{array}{lrcl} \textit{E1:} & \alpha x_1^{\alpha-1} x_2^\beta & = & \lambda p_1 \\ \textit{E2:} & \beta x_1^\alpha x_2^{\beta-1} & = & \lambda p_2 \\ \textit{E3:} & p_1 x_1 + p_2 x_2 & = & c \end{array}$$

- *E1/E2*

$$\frac{\alpha x_1^{\alpha-1} x_2^\beta}{\beta x_1^\alpha x_2^{\beta-1}} \;=\; \frac{\lambda p_1}{\lambda p_2} \;\Leftrightarrow\; \frac{\alpha x_2}{\beta x_1} \;=\; \frac{p_1}{p_2} \;\Leftrightarrow\; x_2 \;=\; \frac{p_1}{p_2}\frac{\beta}{\alpha} x_1$$

- $x_2$ *in E3*

$$p_1 x_1 + p_2 x_2 \;=\; c \;\Leftrightarrow\; p_1 x_1 + p_2 \left( \frac{p_1}{p_2}\frac{\beta}{\alpha} x_1 \right) \;=\; c \;\Leftrightarrow\; x_1^* \;=\; \frac{c\alpha}{p_1(\alpha+\beta)}$$

- $x_1$ *in* $x_2$

$$x_2^* \;=\; \frac{p_1}{p_2}\frac{\beta}{\alpha} x_1 \;=\; \frac{p_1}{p_2}\frac{\beta}{\alpha}\frac{c\alpha}{p_1(\alpha+\beta)} \;=\; \frac{c\beta}{p_2(\alpha+\beta)}$$

- $x_1^*$ *and* $x_2^*$ *in E1*

$$\lambda^* \;=\; \frac{\alpha \left( \dfrac{c\alpha}{p_1(\alpha+\beta)} \right)^{\alpha-1} \left( \dfrac{c\beta}{p_2(\alpha+\beta)} \right)^{\beta}}{p_1} \;=\; \frac{\alpha^\alpha \beta^\beta c^{\alpha+\beta-1}}{p_1^\alpha p_2^\beta (\alpha+\beta)^{\alpha+\beta-1}}$$

- *Hesse matrix of L with respect to* $\mathbf{x}$

$$\nabla^2_{\mathbf{x}} L(\mathbf{x}) \;=\; \begin{pmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^\beta & \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} \\ \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} & \beta(\beta-1) x_1^\alpha x_2^{\beta-2} \end{pmatrix}$$

- *If* $\nabla^2_{\mathbf{x}} L(\mathbf{x})$ *is positive definite (for all* $x_1, x_2 > 0$*) then L is concave and* $\mathbf{x}^* = (x_1^*, x_2^*)$ *solves the maximization problem.*

  *Is* $\nabla^2_{\mathbf{x}} L(\mathbf{x})$ *positive definite?*

## 3.3 $D = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{g}(\mathbf{x}) \leq \mathbf{c}\}$

Given the following optimization problem:

$$\max \qquad y = f(x_1, x_2, \ldots, x_n) = f(\mathbf{x})$$

$$\text{subject to} \qquad \begin{cases} g_1(x_1, x_2, \ldots, x_n) = g_1(\mathbf{x}) \leq c_1 \\ g_2(x_1, x_2, \ldots, x_n) = g_2(\mathbf{x}) \leq c_2 \\ \ldots \\ g_m(x_1, x_2, \ldots, x_n) = g_m(\mathbf{x}) \leq c_m \end{cases}$$

**Definition 3.3** *Let $\mathbf{x}^*$ be the solution of the maximization problem. The constraint $g_i(\mathbf{x}) \leq c_i$ is called*

- *binding (or activ) at $\mathbf{x}^*$, if $g_i(\mathbf{x}^*) = c_i$ and*
- *not binding (or inactiv) at $\mathbf{x}^*$, if $g_i(\mathbf{x}^*) < c_i$.*

**Theorem 3.4** *Suppose that*

- *$f, g_1, \ldots, g_m$ are defined on a set $S \subset \mathbb{R}^n$*
- *$\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ is an interior point of $S$ that solves the maximization problem*
- *$f, g_1, \ldots, g_m$ are continuously partial differentiable in a ball around $\mathbf{x}^*$*
- *the constraints are ordered in such a way, that the first $m_0$ constraints are binding at $\mathbf{x}^*$ and all the remaining $m - m_0$ constraints are not binding,*
- *the Jacobi-matrix of the binding constraint functions*

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}^*) & \ldots & \frac{\partial g_1}{\partial x_n}(\mathbf{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{m_0}}{\partial x_1}(\mathbf{x}^*) & \ldots & \frac{\partial g_{m_0}}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

  *has rank $m_0$ in $\mathbf{x} = \mathbf{x}^*$.*

**Necessary condition**

*Then there exist unique real numbers $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_m^*)$ such that*

1. *$L_{x_1}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0, \ldots, L_{x_n}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0,$*
2. *$\lambda_1^* \geq 0, \ldots, \lambda_m^* \geq 0,$*
3. *$\lambda_1^* \left[g_1(\mathbf{x}^*) - c_1\right] = 0, \ldots, \lambda_m^* \left[g_m(\mathbf{x}^*) - c_m\right] = 0$ and*
4. *$g_1(\mathbf{x}^*) \leq c_1, \ldots, g_m(\mathbf{x}^*) \leq c_m.$*

**Proof:**

**Necessary condition** We study the optimal value function

$$f^*(\mathbf{c}) \;=\; \max\{f(\mathbf{x}) \mid \mathbf{g}(\mathbf{x}) \le \mathbf{c}\}$$

These value function must be nondecreasing in each variable $c_1, \ldots, c_m$. This is because as $c_j$ increases with all other variables held fixed, the admissible set becomes larger; hence $f^*(\mathbf{c})$ can not decrease.

In the following argument we **assume that** $f^*(\mathbf{c})$ **is differentiable.**

Fix a vector $\mathbf{c}^*$ and let $\mathbf{x}^*$ be the corresponding optimal solution. Then $f(\mathbf{x}^*) = f^*(\mathbf{c}^*)$. For any $\mathbf{x}$ we have $f(\mathbf{x}) \le f^*(\mathbf{g}(\mathbf{x}))$ because $\mathbf{x}$ obviously satisfies the constraints when each $c_j^*$ is replaced by $g_j(\mathbf{x})$.

But then

$$f^*(\mathbf{g}(\mathbf{x})) \;\le\; f^*(\mathbf{g}(\mathbf{x}) + \underbrace{\mathbf{c}^* - \mathbf{g}(\mathbf{x}^*)}_{\ge 0})$$

since $\mathbf{g}(\mathbf{x}^*) \le \mathbf{c}^*$ and $f^*$ is nondecreasing.

Hence

$$\phi(\mathbf{x}) \;:=\; f(\mathbf{x}) - f^*(\underbrace{\mathbf{g}(\mathbf{x}) + \mathbf{c}^* - \mathbf{g}(\mathbf{x}^*)}_{=:\mathbf{u}(\mathbf{x})}) \;\le\; 0$$

for all $\mathbf{x}$ and since $\phi(\mathbf{x}^*) = 0$, $\phi(\mathbf{x})$ has a maximum in $\mathbf{x} = \mathbf{x}^*$, so

$$\begin{aligned} 0 \;=\; \frac{\partial \phi}{\partial x_i}(\mathbf{x}^*) \;&=\; \frac{\partial f}{\partial x_i}(\mathbf{x}^*) - \sum_{j=1}^{m} \frac{\partial f^*}{\partial u_j}(\mathbf{u}(\mathbf{x}^*))\, \frac{\partial u_j}{\partial x_i}(\mathbf{x}^*) \\ &=\; \frac{\partial f}{\partial x_i}(\mathbf{x}^*) - \sum_{j=1}^{m} \frac{\partial f^*}{\partial u_j}(\mathbf{c}^*)\, \frac{\partial g_j}{\partial x_i}(\mathbf{x}^*) \end{aligned}$$

Since $f^*$ is nondecreasing, we have

$$\lambda_j^* \;:=\; \frac{\partial f^*}{\partial c_j}(\mathbf{c}) \;\ge\; 0.$$

and we should (but will not) prove that if $g_j(\mathbf{x}^*) < c_j^*$ then $\lambda_j^* = 0$. $\square$