Differentiable processes and differential equations

Keywords: differential processes, differential equations, systems of differential equations, autonomous systems, phase plane analysis, equilibrium point, linear systems (of differential equations) with constant coefficients

Contents

1 Introduction

Processes and ordinary differential equations The theory of ordinary differential equations is one of the basic tools of mathematical science. The theory allows us to study all kinds of evolutionary processes with the properties of

- determinacy,
- finite-dimensionality and
- differentiability.

Definition 1.1 *A process is said to be deterministic if its entire future course and its entire past are uniquely determined by its state at the present time. The set of all possible states of a process is called its phase space.*

Example 1.1

- *Classical mechanical systems are deterministic and the phase space is the set of instantaneous positions and velocities of all particles of the system.*
- *The motion of particles in quantum mechanics is not described by a deterministic process.*
- *Heat propagation is a semi-deterministic process.*

Definition 1.2 *A process is said to be finite-dimensional if its phase space is finite-dimensional (if the number of parameters required to describe its state is finite).*

Example 1.2 *A mechanical system of n particles is finite-dimensional. The dimension of the phase space is $6n$: $\begin{pmatrix} x_i & y_i & z_i \end{pmatrix}$ for the position of a particle and $\begin{pmatrix} \dot{x}_i & \dot{y}_i & \dot{z}_i \end{pmatrix}$ for the velocity of a particle in space.*

Definition 1.3 *A process is said to be differentiable if its phase space has the structure of a differentiable manifold and if its change of state with time is described by differentiable functions.*

You do **not** need to know, what a differentiable manifold is. You should understand that if you have a differentiable process, then all (finitely many) parameters, describing the state of the system completely, change differentiably in time.

We will focus our attention on deterministic, **2-dimensional** and differentiable processes, which can be described by two first-order differential equations in two variables.

Definition 1.4 *Let f, g be functions in 3 variables. A* <u>*normal system*</u> *of 2 first-order differential equations in 2 variables takes the form*

$$\begin{aligned}\frac{dx(t)}{dt} &= \dot{x} = f(x, y, t)\\ \frac{dy(t)}{dt} &= \dot{y} = g(x, y, t)\end{aligned}$$

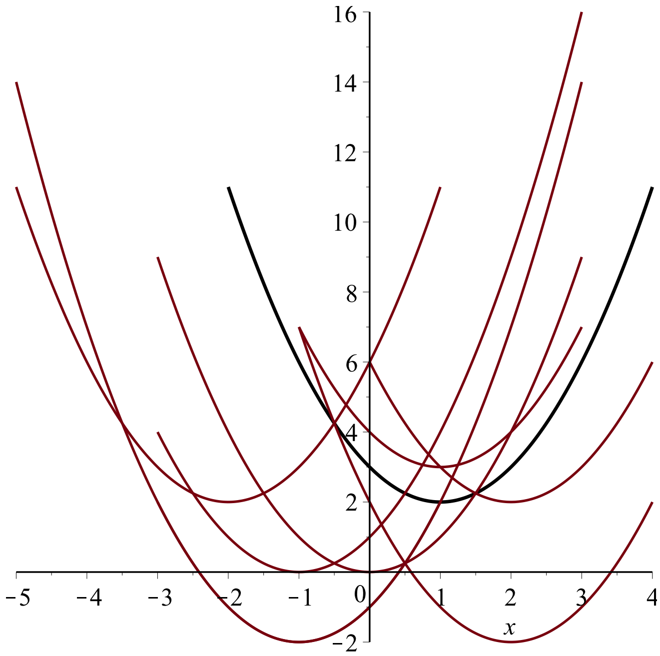
A solution is a pair of differentiable functions $(x(t), y(t))$, defined on some interval I, that satisfies both equations.

Theorem 1.1 *If f, g, f_x, f_y, g_x, g_y are continuous then we have the follwing fact:*
If $t_0 \in I$ and $x_0, y_0 \in \mathbb{R}$ there is **one and only one** *pair of functions $(x(t), y(t))$ that satisfies the two equations and $x(t_0) = x_0$ and $y(t_0) = y_0$.*

Example 1.3

$$\begin{aligned}\dot{x} &= 1\\ \dot{y} &= 2t\end{aligned}$$

Integrate each equation directly: $x(t) = t + C_1$ and $y(t) = t^2 + C_2$ or $y = (x - C_1)^2 + C_2$.
With the initial condition $x_0 = 1$ and $y_0 = 2$ we get the solution $x(t) = t + 1$ and $y(t) = t^2 + 2$ or $y = (x - 1)^2 + 2$.



Example 1.4 *We look at the so-called* **(uncontrolled) predator-prey system** *described by the so-called* **predator-prey equations**

$$\begin{aligned}\dot{x} &= \alpha\, x - \beta\, x\, y\\ \dot{y} &= \delta\, x\, y - \gamma\, y\end{aligned}$$

with

- $x = x(t)$ *be the number of prey,*
- $y = y(t)$ *the number of predator and*
- $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ *parameters, describing the interaction of the two species.*

  Short explanation:

  - αx*:*
    The prey is assumed to have an unlimited food supply and reproduces exponentially unless subject to predation.
  - $-\beta xy$
    The rate of predation upon the prey is assumed to be proportional to the rate at which the predators and the prey meet.
  - δxy
    The growth of the predator population is proportional to the rate at which the predators and the prey meet.
  - $-\gamma y$
    The predators are assumed to decay exponentially in the absence of prey.

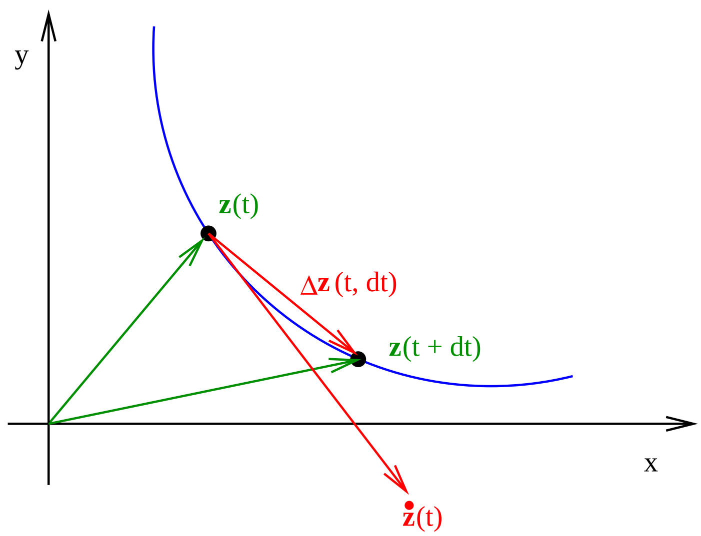
We use the following notations:

$$\mathbf{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad \mathbf{z}(t+dt) = \begin{pmatrix} x(t+dt) \\ y(t+dt) \end{pmatrix}$$

$$\Delta\mathbf{z}(t,dt) = \mathbf{z}(t+dt) - \mathbf{z}(t) = \begin{pmatrix} x(t+dt) - x(t) \\ y(t+dt) - y(t) \end{pmatrix}$$

$$\dot{\mathbf{z}}(t) = \lim_{dt\to 0} \frac{\Delta\mathbf{z}(t,dt)}{dt} = \lim_{dt\to 0} \frac{\mathbf{z}(t+dt) - \mathbf{z}(t)}{dt} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$$

If $\mathbf{z}(t)$ is a solution of the system, then the points $\mathbf{z}(t)$ trace out a curve K in the xy-plane. The vector $\dot{\mathbf{z}}(t)$, which describes how quickly $x(t)$ and $y(t)$ change when t is changed, is the tangent vector to the curve K in the point $(x(t), y(t))$.



How can we find a general solution of the system

$$\begin{aligned} \dot{x} &= f(t,x,y) \\ \dot{y} &= g(t,x,y) \end{aligned}$$

We can **not** expect exact methods that work in complete generality! One important method: Reduce the system to a (single) second-order differential equation in the following way.

- Use the first equation to express y as a function $y = h(t, x, \dot{x})$,
- Differentiate this equation with respect to t and
- Substitute the expressions for y and $\dot{y}$ in the second equation.

Example 1.5 *Find the general solution of the system*

$$\begin{array}{rcl} (I)\ \dot{x} & = & 2x + e^t y - e^t \\ (II)\ \dot{y} & = & 4e^{-t}x + y \end{array}$$

Solution:

- $(I) \ \leftrightarrow \ y = e^{-t}\dot{x} - e^{-t}2x + 1 \ \rightarrow \ \dot{y} = -e^{-t}\dot{x} + e^{-1}\ddot{x} + e^{-t}2x - e^{-t}2\dot{x}$
- *in* (II) $\dot{y} = 4e^{-t}x + y \ \leftrightarrow \ 0 = \ddot{x} - 4\dot{x} - e^t$
- *General solution:* $x(t) = C_1 + C_2 e^{4t} + \frac{1}{3}e^t$
- $y(t) = e^{-t}\dot{x} - e^{-t}2x + 1 = \ldots$

2 Phase plane analysis

We will indicate how geometric arguments can help to understand the **structure** of the solutions of an autonomous system.

Definition 2.1 *A system of differential equations is called* <u>*autonomous*</u> *(or* <u>*time independent*</u>*), if f and g do not depend on t explicitely.*

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}$$

Autonomous systems have nice properties:

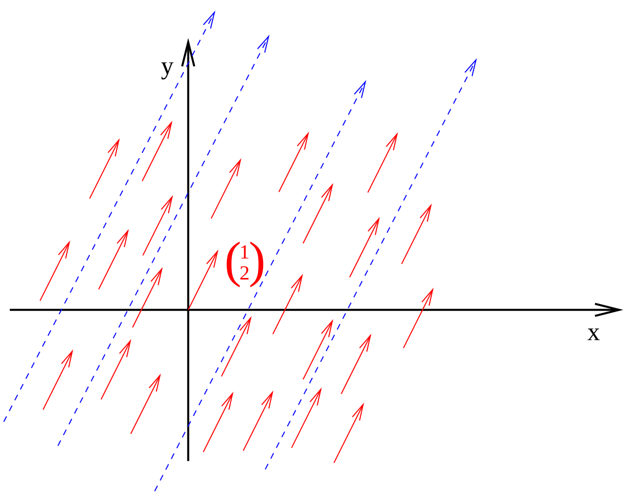
- A solution $(x(t), y(t))$ describes a curve or path in the xy-plane.
- The vector $\dot{\mathbf{z}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$ is uniquely determined at each point $(x(t), y(t))$ (independent of t) and

$$\dot{\mathbf{z}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

- Two solution pathes do not intersect in the xy-plane.

To illustrate the dynamics of the autonomous system, we can, in principle, draw a vector $\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$ at each point (x,y) (<u>vector field</u>) and then we can draw solution curves for the system (<u>phase diagram</u>).

Example 2.1 *For the system $\dot{x} = 1; \dot{y} = 2$ we have $\dot{\mathbf{z}}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for all (x,y) and the solution curves (blue) point in this direction and must be straight lines.*



Definition 2.2 *A point (a, b) in the xy-plane with $f(a, b) = g(a, b) = 0$ is called an equilibrium point or stationary point. The two curves defined by $f(x, y) = 0$ and $g(x, y) = 0$ are called the nullclines of the system.*

At an equilibrium point E we have $\dot{x} = \dot{y} = 0$ if the system is in E. Hence the system will always be (and always was) at E. To draw the phase diagram, begin by drawing the two nullclines.

- at each point on $f(x, y) = 0 \quad \rightarrow \quad \dot{x} = 0$ (vertical velocity vector)
- at each point on $g(x, y) = 0 \quad \rightarrow \quad \dot{y} = 0$ (horizontal velocity vector)

3 Linear systems with constant coefficients

Consider the linear system with constant coefficients

$$\begin{aligned} \dot{x} &= a_{11}x + a_{12}y + b_1(t) && \diamond_1 \\ \dot{y} &= a_{21}x + a_{22}y + b_2(t) && \diamond_2 \end{aligned}$$

Transformation ($a_{12} \neq 0$):

1. Differentiating $\diamond_1$ w.r.t. t, then substituting $\dot{y}$ from $\diamond_2$:
   $\ddot{x} = a_{11}\dot{x} + a_{12}(a_{21}x + a_{22}y + b_2(t)) + \dot{b}_1(t)$
2. Substituting $a_{12}y = \dot{x} - a_{11}x - b_1(t)$ (from $\diamond_1$) :
   $\ddot{x} - (a_{11} + a_{22})\dot{x} + (a_{11}a_{22} - a_{12}a_{21})x = a_{12}b_2(t) - a_{22}b_1(t) + \dot{b}_1(t)$
   Second-order differential equation with constant coefficients with general solution of type $x(t) = C_1x_1 + C_2x_2 + x^p$ with $C_1, C_2 \in \mathbb{R}$
3. The solution for $y(t)$ can be found from $a_{12}y = \dot{x} - a_{11}x - b_1(t)$ and depends on the same two constants C_1, C_2.

With $b_1 = b_2 = 0$ the system reduces to the homogeneous system

$$\begin{aligned} \dot{x} &= a_{11}x + a_{12}y \\ \dot{y} &= a_{21}x + a_{22}y \end{aligned} \quad \text{or} \quad \underbrace{\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}}_{=\dot{\mathbf{z}}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{=\mathbf{A}} \underbrace{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}}_{=\mathbf{z}}$$

Theorem 3.1 (Solution of $\dot{\mathbf{z}} = \mathbf{A}\,\mathbf{z}$) *The set*

$$\mathcal{L} = \{ \mathbf{z} \mid \dot{\mathbf{z}} = \mathbf{A}\,\mathbf{z}, t \in I \}$$

is an 2-dimensional vector space. A fundamental system (or fundamental matrix) of the homogeneous system $\dot{\mathbf{z}} = \mathbf{A}\,\mathbf{z}$ is a basis $\mathbf{F}(t) = (\,\mathbf{z}_1(t), \mathbf{z}_2(t)\,)$ (written as a matrix) of $\mathcal{L}$. In this case, the general solution can be written as

$$\mathbf{z}(t) = c_1\mathbf{z}_1(t) + c_2\mathbf{z}_2(t).$$

Definition 3.1 *The determinant $W(t) = \det \mathbf{F}(t)$ of a solution matrix $\mathbf{F}(t)$ of $\dot{\mathbf{z}} = \mathbf{A}\,\mathbf{z}$ is called Wronski-determinant.*

Remarks 3.1

- *The 2 functions* $\mathbf{z}_1, \mathbf{z}_2 : I \to \mathbb{R}^2$ *are* <u>*linearly independent*</u> *if and only if* $\alpha_1\mathbf{z}_1 + \alpha_2\mathbf{z}_2 = \mathbf{0}$ *for all* $t \in I$ *always implies* $\alpha_1 = \alpha_2 = 0$.
- *Let* $\mathbf{z}_1, \mathbf{z}_2$ *be solutions of the system* $\dot{\mathbf{z}} = \mathbf{A}\ \mathbf{z}$. *Then these functions are linearly independent if and only if* $W(t_0) = \det \mathbf{F}(t_0) \neq 0$ *for at least one* $t_0 \in I$.
- *The solution of* $\dot{\mathbf{z}} = \mathbf{A}\ \mathbf{z}$ *can be written as*
$$\mathbf{z}(t) \quad = \quad \mathbf{F}(t)\mathbf{c}.$$
- *The solution of the initial value problem* $\dot{\mathbf{z}} = \mathbf{A}\ \mathbf{z}$ *and* $\mathbf{z}(t_0) = \mathbf{z}_0$ *can be written as*
$$\mathbf{z}(t) \quad = \quad \mathbf{F}(t)\,\underbrace{\mathbf{F}(t_0)^{-1}\mathbf{z}_0}_{=\mathbf{c}}.$$
- *If* λ *is an eigenvalue of* $\mathbf{A}$ *with associated eigenvector* $\mathbf{v}$, *then* $\mathbf{z}(t) = \mathbf{v}e^{\lambda t}$ *is a solution of* $\dot{\mathbf{z}} = \mathbf{A}\ \mathbf{z}$, *because*
$$\mathbf{A}\ \mathbf{z} \quad = \quad \mathbf{A}\ \mathbf{v}e^{\lambda t} = \lambda\mathbf{v}e^{\lambda t} = \dot{\mathbf{z}}.$$
- *If* $\mathbf{A}$ *has 2 different real eigenvalues* λ_1, λ_2 *with associated eigenvectors* $\mathbf{v}_1, \mathbf{v}_2$ *then*
$$\mathbf{z}_1(t) \;=\; \mathbf{v}_1 e^{\lambda_1 t} \quad \textit{and} \quad \mathbf{z}_2(t) \;=\; \mathbf{v}_2 e^{\lambda_2 t}$$
are two basis solutions of $\dot{\mathbf{z}} = \mathbf{A}\ \mathbf{z}$.

Example 3.1 *The matrix* $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ *has the eigenvalues and eigenvectos*

$$\lambda_1 = 2 \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3 \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Hence the system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ *has the two basis solutions*

$$\mathbf{z}_1(t) \;=\; e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \textit{and} \quad \mathbf{z}_2(t) \;=\; e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and the general solution can be written as

$$\mathbf{z}(t) \;=\; A\,e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + B\,e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Theorem 3.2 *Let* $\mathbf{A} \in \mathbb{R}^{2\times 2}$ *be a square matrix. We already know: There exists a basis of* $\mathbb{C}^2$ *consisting of generalized eigenvectors of* $\mathbf{A}$*.*

For all generalized eigenvectors $\mathbf{v}$ *of degree* l *(*$= 1$ *or* 2*) associated to the eigenvalue* λ *of* $\mathbf{A}$*, one basis solution of* $\dot{\mathbf{z}} = \mathbf{A}\ \mathbf{z}$ *is given by:*

$$\mathbf{z}(t) = \begin{cases} e^{\lambda t}\ \mathbf{v} & \text{if } l = 1 \\ e^{\lambda t}\ [\mathbf{v} + t(\mathbf{A} - \lambda\mathbf{I})\ \mathbf{v}] & \text{if } l = 2 \end{cases} .$$

Proof:

- If $\mathbf{v}$ is an eigenvector ($l = 1$) then we already know that $e^{\lambda t}\ \mathbf{v}$ is a solution.
- If $\mathbf{v}$ is a generalized eigenvector of degree $l = 2$ then we know that $(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v} = \mathbf{0}$ or $\mathbf{A}(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \lambda(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}$.

  For $\mathbf{z}(t) = e^{\lambda t}\ [\mathbf{v} + t(\mathbf{A} - \lambda\mathbf{I})\ \mathbf{v}]$ we get

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \lambda e^{\lambda t}\ [\mathbf{v} + t(\mathbf{A} - \lambda\mathbf{I})\ \mathbf{v}] + e^{\lambda t}(\mathbf{A} - \lambda\mathbf{I})\ \mathbf{v} \\ &= e^{\lambda t}\ [\mathbf{A}\ \mathbf{v} + t\lambda(\mathbf{A} - \lambda\mathbf{I})\ \mathbf{v}] \end{aligned}$$

  and

$$\begin{aligned} \mathbf{A}\mathbf{z}(t) &= e^{\lambda t}\ [\mathbf{A}\mathbf{v} + t\mathbf{A}(\mathbf{A} - \lambda\mathbf{I})\ \mathbf{v}] \\ &= e^{\lambda t}\ [\mathbf{A}\ \mathbf{v} + t\lambda(\mathbf{A} - \lambda\mathbf{I})\ \mathbf{v}] \,. \end{aligned}$$

□

4 *Classification of all linear systems in plane*

We would like to investigate the systems

$$\dot{\mathbf{z}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{=\,\mathbf{A}} \mathbf{z}$$

Each solution can be viewed as a curve in the (x, y)-plane. The right side of the equation defines a vector field in the plane and the solutions have to follow this vector field. We now try to understand the structure of the solutions and the dependence of the solutions from the matrix $\mathbf{A}$. We determine the eigenvalues, the generalized eigenvectors and the associated basis solutions $\mathbf{z}_1(t)$ and $\mathbf{z}_2(t)$. Then all solutions are linear combinations

$$\mathbf{z}(t) = c_1\, \mathbf{z}_1(t) + c_2\, \mathbf{z}_2(t)$$

of the basis solutions. Of course, the basis solutions depend on the two eigenvalues of $\mathbf{A}$ and there are 14 different cases (divided into 4 groups A, B, C and D). Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the eigenvalues of $\mathbf{A}$, this means the solutions of the characteristic equation

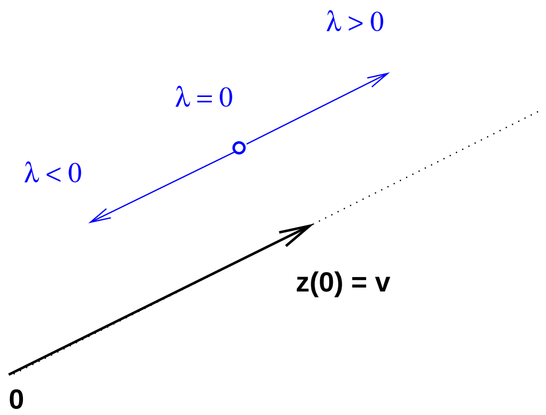
$$p_{\mathbf{A}}(\lambda) = \lambda^2 - tr\mathbf{A}\,\lambda + \det \mathbf{A} = (\lambda_1 - \lambda)\,(\lambda_2 - \lambda) = 0.$$

A) $\underline{\lambda_1, \lambda_2 \in \mathbb{R} \text{ and } \lambda_1 \neq \lambda_2}$

There are two linearly independent (real) eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ associated to λ_1 and λ_2 and the two basic solutions are

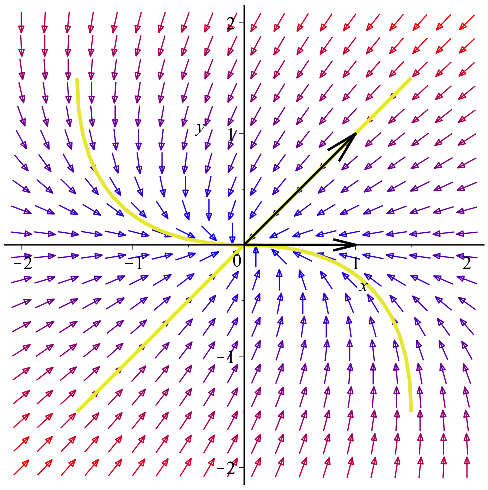
$$\mathbf{z}_1(t) = e^{\lambda_1 t}\, \mathbf{v}_1 \qquad \text{and} \qquad \mathbf{z}_2(t) = e^{\lambda_2 t}\, \mathbf{v}_2.$$

A solution path $\mathbf{z}(t) = e^{\lambda t}\, \mathbf{v}$ starts in $\mathbf{z}(0) = \mathbf{v}$ and goes along a straight line in the direction of $\mathbf{v}$ (if $\lambda > 0$) or $-\mathbf{v}$ (if $\lambda < 0$). If $\lambda = 0$ then $\mathbf{z}(t) = e^{0t}\, \mathbf{v} = \mathbf{v}$ for all t.

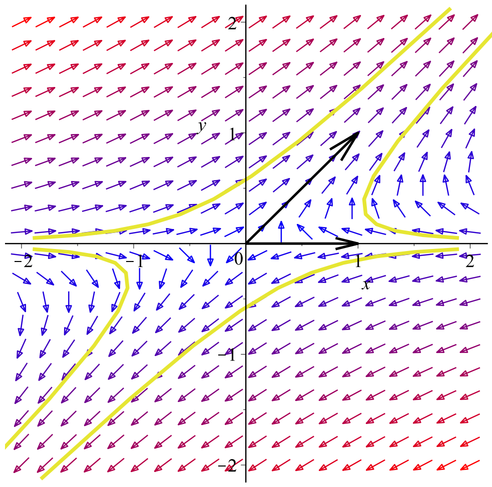


All solutions of the systems are linear combinations $\mathbf{z}(t) = c_1\, \mathbf{z}_1(t) + c_2\, \mathbf{z}_2(t)$. There are 5 cases, depending on the signs of the eigenvalues.

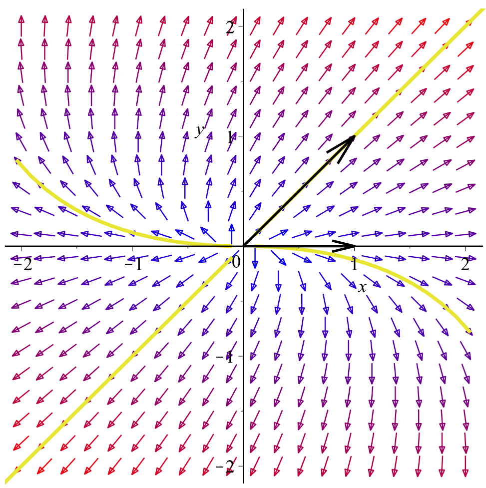
A1) $\lambda_1 < \lambda_2 < 0$



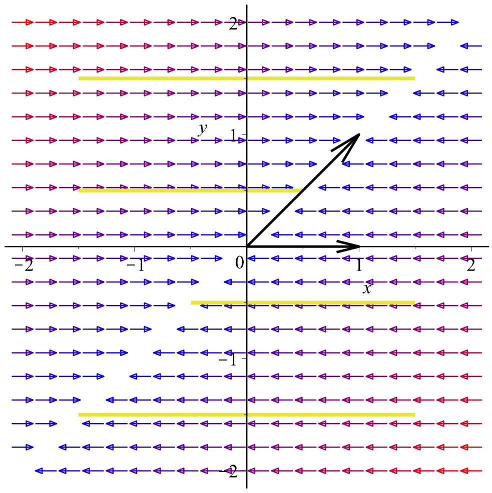
A2) $\lambda_1 < 0 < \lambda_2$



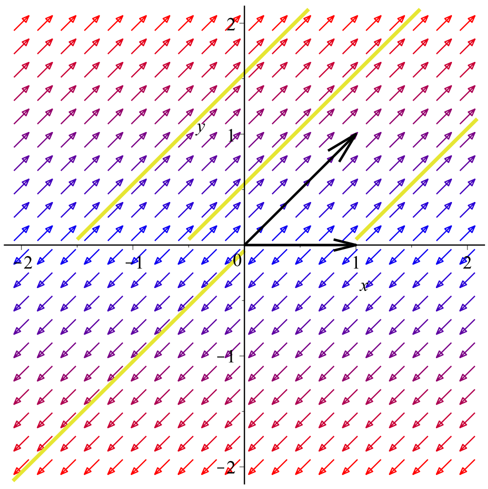
A3) $0 < \lambda_1 < \lambda_2$



A4) $\lambda_1 < \lambda_2 = 0$



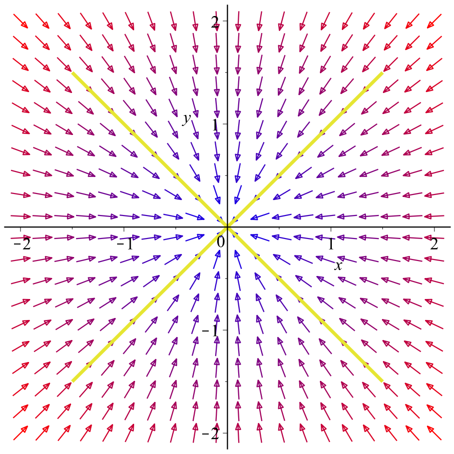
A5) $\lambda_1 = 0 < \lambda_2$



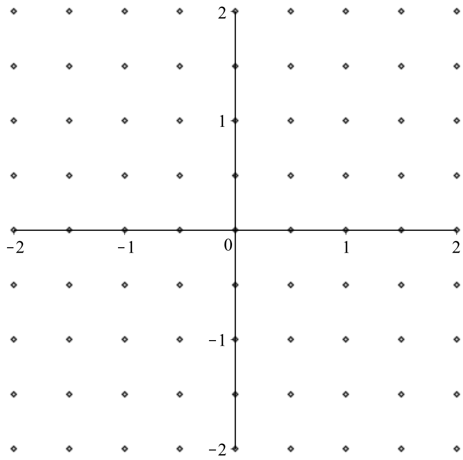
B) $\underline{\lambda = \lambda_1 = \lambda_2 \in \mathbb{R} \text{ and } \dim V(\lambda) = 2}$

We have $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ for **all** $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{A} = \lambda\,\mathbf{I}$. For **all** $\mathbf{c} \in \mathbb{R}^2$ $\mathbf{z}(t) = e^{\mathbf{A}t}\,\mathbf{c} = e^{\lambda t}\,\mathbf{c}$ is a solution. There are 3 different cases.

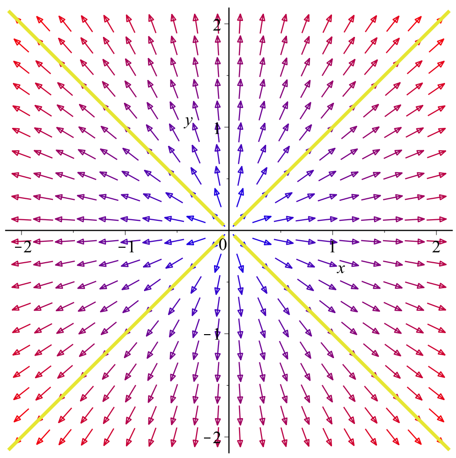
B1) $\lambda < 0$



B2) $\lambda = 0$ Then $\mathbf{z}(t) = \mathbf{c}$ is a constant solution for all $\mathbf{c}$.



B3) $\lambda > 0$



C) $\underline{\lambda = \lambda_1 = \lambda_2 \in \mathbb{R} \text{ and } \dim V(\lambda) = 1}$

If $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ associated to λ and $\mathbf{u}$ is a generalized eigenvector of degree 2 ($(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{v}$), then there are the two basis solutions

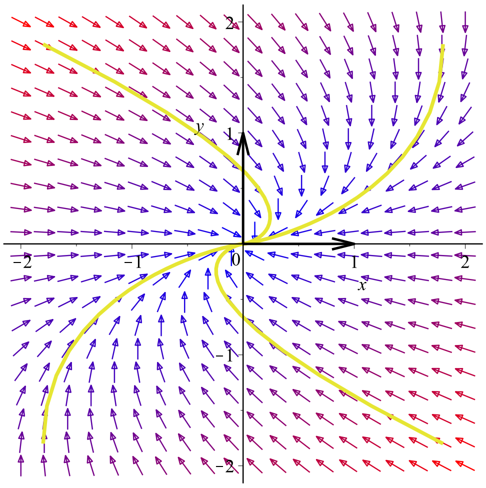
$$\mathbf{z}_1(t) \;=\; e^{\lambda t}\,\mathbf{v} \qquad \text{and} \qquad \mathbf{z}_2(t) \;=\; e^{\lambda t}\,[\,\mathbf{u} + t\,\underbrace{(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}}_{=\mathbf{v}}\,]$$

The general solution is

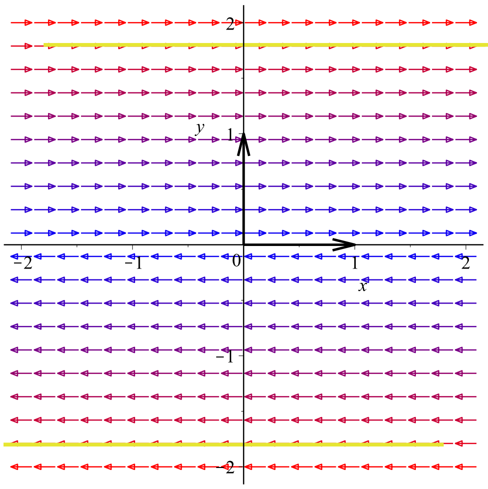
$$\mathbf{z}(t) \;=\; c_1\,e^{\lambda t}\,\mathbf{v} + c_2\,e^{\lambda t}\,[\,\mathbf{u} + t\mathbf{v}\,] \;=\; e^{\lambda t}\,[\,(c_1 + c_2 t)\mathbf{v} + c_2\mathbf{u}\,]$$

We see that for $t \to \pm\infty$ the direction of the solution $\mathbf{z}(t)$ tends to the directions $\pm\mathbf{v}$. There are 3 different cases.

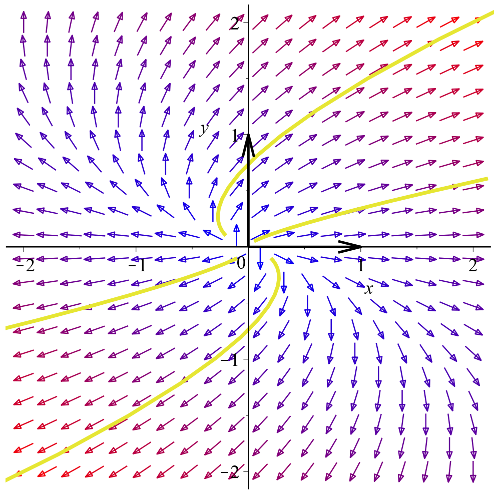
C1) $\lambda < 0$



C2) $\lambda = 0$



C3) $\lambda > 0$



D) $\underline{\lambda_2 = \overline{\lambda_1} \in \mathbb{C} - \mathbb{R}}$

Let $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \overline{\lambda_1} = \alpha - i\beta$. It is easy to see that if $\mathbf{A}\mathbf{v} = \lambda_1 \mathbf{v}$ then $\mathbf{A}\overline{\mathbf{v}} = \lambda_2 \overline{\mathbf{v}}$, because $\mathbf{A}$ is a real matrix. This means that if $\mathbf{v} \in \mathbb{C}^2$ is a (complex) eigenvector of $\mathbf{A}$ associated to λ_1 then $\overline{\mathbf{v}} \in \mathbb{C}^2$ is a (complex) eigenvector of $\mathbf{A}$ associated to $\lambda_2 = \overline{\lambda_1}$.

The complex basis solutions are (with $\mathbf{a} = \Re(\mathbf{v})$ and $\mathbf{b} = \Im(\mathbf{v})$)

$$\begin{aligned}
\mathbf{x}_1(t) &= e^{\lambda_1 t}\, \mathbf{v} \;=\; e^{(\alpha+i\beta)t}\, (\mathbf{a} + i\mathbf{b}) \;=\; e^{\alpha t} e^{i\beta t}\, (\mathbf{a} + i\mathbf{b}) \\
&= e^{\alpha t} (\cos\beta t + i \sin\beta t)\, (\mathbf{a} + i\mathbf{b}) \\
&= e^{\alpha t}\, [\, (\cos\beta t \cdot \mathbf{a} - \sin\beta t \cdot \mathbf{b}) + i(\sin\beta t \cdot \mathbf{a} + \cos\beta t \cdot \mathbf{b})\,]
\end{aligned}$$

$$\mathbf{x}_2(t) = e^{\alpha t}\, [\, (\cos\beta t \cdot \mathbf{a} + \sin\beta t \cdot \mathbf{b}) - i(\sin\beta t \cdot \mathbf{a} + \cos\beta t \cdot \mathbf{b})\,]$$

It is easy to prove that

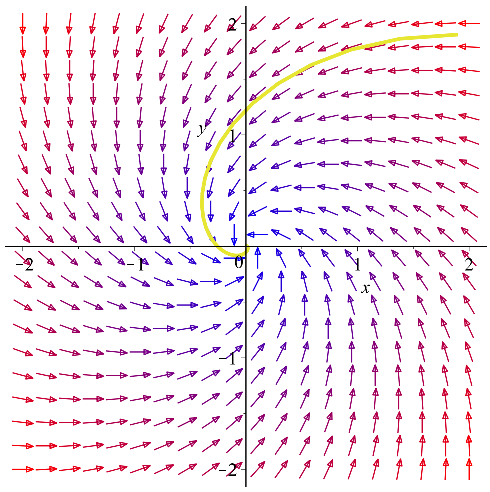
$$\begin{aligned}
\mathbf{z}_1(t) &= \Re\,(\mathbf{x}_1(t)) \;=\; e^{\alpha t}\, (\cos\beta t \cdot \mathbf{a} - \sin\beta t \cdot \mathbf{b}) \\
\mathbf{z}_2(t) &= \Im\,(\mathbf{x}_1(t)) \;=\; e^{\alpha t}\, (\sin\beta t \cdot \mathbf{a} + \cos\beta t \cdot \mathbf{b})
\end{aligned}$$

are two linearly independent (over $\mathbb{R}$) real solutions of the system $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$. The general real solution can be written as

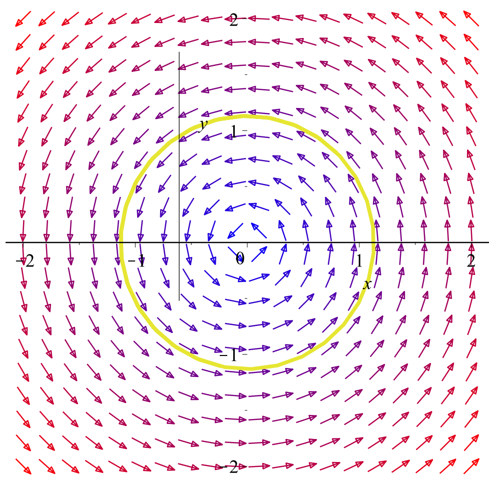
$$\begin{aligned}
\mathbf{z}(t) &= c_1 \mathbf{z}_1(t) + c_2 \mathbf{z}_2(t) \\
&= e^{\alpha t}\, [\, (c_1 \cos\beta t + c_2 \sin\beta t) \cdot \mathbf{a} + (-c_1 \sin\beta t + c_2 \cos\beta t) \cdot \mathbf{b}\,]\, .
\end{aligned}$$

There are 3 different cases.

D1) $\alpha < 0$



D2) $\alpha = 0$



D3) $\alpha > 0$