Optimal control problems

Keywords: functional, increment, calculus of variations, fundamental theorem of calculus of variations, Euler equation, optimal control problems, performance measure, Pontryagin's Minimimum Principle

Contents

1 Functionals and calculus of variations

1.1 Functionals and Increments

Definition 1.1 *A functional J is a rule of correspondence that assigns to each function $\mathbf{u} = \mathbf{u}(t)$ in a certain class (of functions) Ω a unique real number. Ω is called the domain of J and the set of real numbers associated with the functions in Ω is called the range of the functional J.*

A functional J is called linear, if and only if

1. *(**principle of homogeneity**) $J(\alpha \mathbf{u}) \;=\; \alpha J(\mathbf{u})$*
   for all $\mathbf{u} \in \Omega$ and all $\alpha \in \mathbb{R}$ with $\alpha \mathbf{u} \in \Omega$
2. *(**principle of additivity**) $J(\mathbf{u}^{(1)} + \mathbf{u}^{(2)}) \;=\; J(\mathbf{u}^{(1)}) + J(\mathbf{u}^{(2)})$*
   for all $\mathbf{u}^{(1)}, \mathbf{u}^{(2)} \in \Omega$ with $\mathbf{u}^{(1)} + \mathbf{u}^{(2)} \in \Omega$

When two points are said to be close to one another, a geometric interpretation immediately springs to mind. But what do we mean if we say two functions are close to one another? To give a precise meaning to the term "close" we introduce the concept of a norm.

Definition 1.2 *A norm $||\mathbf{u}||$ of a function $\mathbf{u} = \mathbf{u}(t)$ is a rule of correspondence that assigns to each function $\mathbf{u} \in \Omega$ defined for $t \in [t_0, t_f]$ a real number such that*

1. *$||\mathbf{u}|| \geq 0$ and $||\mathbf{u}|| = 0$ if and only if $\mathbf{u}(t) = \mathbf{0}$ for all t;*
2. *$||\alpha \mathbf{u}|| \;=\; |\alpha| \cdot ||\mathbf{u}||$ and*
3. *$||\mathbf{u}^{(1)} + \mathbf{u}^{(2)}|| \;\leq\; ||\mathbf{u}^{(1)}|| + ||\mathbf{u}^{(2)}||$*

for all $\mathbf{u}, \mathbf{u}^{(1)}, \mathbf{u}^{(2)} \in \Omega$ and $\alpha \in \mathbb{R}$.

Let $u = u(t)$ be a continuous scalar function defined in the interval $[t_0, t_f]$. Then

$$||u|| \;:=\; \max_{t_0 \leq t \leq t_f} \{|u(t)|\}$$

is a norm, because it satisfies the three properties.

Definition 1.3 *Let $\mathbf{u}$ and $\delta\mathbf{u}$ be functions such that a functional J is defined for $\mathbf{u}, \delta\mathbf{u}$ and $\mathbf{u} + \delta\mathbf{u}$. Then the increment of J, denoted by $\Delta J = \Delta J(\mathbf{u}, \delta\mathbf{u})$ is*

$$\Delta J \;:=\; J(\mathbf{u} + \delta\mathbf{u}) - J(\mathbf{u}).$$

We should write $\Delta J(\mathbf{u}, \delta\mathbf{u})$ to emphasize that the increment depends on the functions $\mathbf{u}$ and $\delta\mathbf{u}$. The function $\delta\mathbf{u}$ is called the variation of the function $\mathbf{u}$.

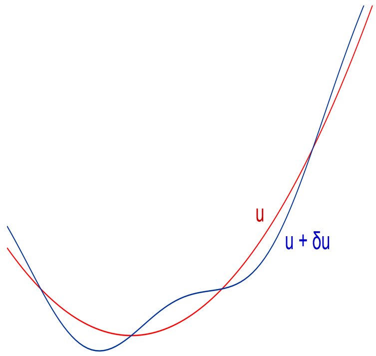
The increment of a functional can be written as

$$\Delta J(\mathbf{u}, \delta\mathbf{u}) \;=\; \delta J(\mathbf{u}, \delta\mathbf{u}) + ||\delta\mathbf{u}|| \cdot g(\mathbf{u}, \delta\mathbf{u})$$

where δJ *is linear in* $\delta\mathbf{u}$. *If*

$$\lim_{||\delta\mathbf{u}||\to 0} g(\mathbf{u}, \delta\mathbf{u}) \;=\; 0$$

then J *is said to be* <u>*differentiable*</u> *in* $\mathbf{u}$ *and* δJ *is the* <u>*variation*</u> *of* J *evaluated at the function* $\mathbf{u}$.



Example 1.1 *Let* $u = u(t)$ *be a continuous scalar function defined for* $t \in [0, 1]$ *and let*

$$J(u) \;:=\; \int_0^1 [u^2 + 2u] \, dt.$$

Then the increment of J *is*

$$\begin{aligned} \Delta J &= J(u + \delta u) - J(u) \\ &= \int_0^1 [(u + \delta u)^2 + 2(u + \delta u)] \, dt - \int_0^1 [u^2 + 2u] \, dt \\ &= \underbrace{\int_0^1 [2u + 2] \, \delta u \, dt}_{=\delta J(u, \delta u)} + \int_0^1 (\delta u)^2 \, dt \end{aligned}$$

Since u *is a continuous function, let* $||\delta u|| = \max_{0\le t\le 1} \{|\delta u(t)|\}$. *Then*

$$\int_0^1 (\delta u)^2 \, dt \;=\; \frac{||\delta u||}{||\delta u||} \int_0^1 (\delta u)^2 \, dt \;=\; ||\delta u|| \underbrace{\int_0^1 \frac{(\delta u)^2}{||\delta u||} \, dt}_{=g(u, \delta u)}$$

and

$$g(u, \delta u) \;=\; \int_0^1 \frac{|\delta u| \cdot |\delta u|}{||\delta u||} \, dt \;\le\; \int_0^1 |\delta u| \, dt \;\longrightarrow\; 0 \quad \textit{if } ||\delta u|| \to 0.$$

1.2 Relative Extrema of functionals and fundamental theorem

Definition 1.4 *A functional J with domain Ω has a relative extremum at $\mathbf{u}^* = \mathbf{u}^*(t)$ if there is an $\epsilon > 0$ such that for all functions $\mathbf{u} \in \Omega$ which satisfy $||\mathbf{u} - \mathbf{u}^*|| < \epsilon$ the increment of J has the same sign.*

- *If $\Delta J = J(\mathbf{u}) - J(\mathbf{u}^*) \geq 0$ then $J(\mathbf{u}^*)$ is a relative minimum.*
- *If $\Delta J = J(\mathbf{u}) - J(\mathbf{u}^*) \leq 0$ then $J(\mathbf{u}^*)$ is a relative maximum.*

If any of these conditions is satisfied for arbitrarily large ϵ then $J(\mathbf{u}^*)$ is a global minimum resp. global maximum. $\mathbf{u}^*$ is called an extremal and $J(\mathbf{u}^*)$ an extremum.

Let $\mathbf{u} = \mathbf{u}(t)$ be a function of t in the class Ω and $J(\mathbf{u})$ be a differentiable functional of $\mathbf{u}$.

Theorem 1.1 (Fundamental Theorem of calculus of Variations I)
Assume that the functions in Ω are not constrained by any boundaries. If $\mathbf{u}^*$ is an extremal of the differentiable functional J, then the variation of J must vanish on $\mathbf{u}^*$, that is for all (admissible) $\delta\mathbf{u}$ with $\mathbf{u}^* + \delta\mathbf{u} \in \Omega$ we have

$$\boxed{\delta J(\mathbf{u}^*, \delta\mathbf{u}) = 0}$$

Proof by contradiction:
Assume $\mathbf{u}^*$ is an extremal and $\delta J(\mathbf{u}^*, \delta\mathbf{u}^{(1)}) \neq 0$.
The increment is

$$\Delta J(\mathbf{u}^*, \delta\mathbf{u}) = \delta J(\mathbf{u}^*, \delta\mathbf{u}) + ||\delta\mathbf{u}|| \cdot g(\mathbf{u}^*, \delta\mathbf{u})$$

and

$$\lim_{||\delta\mathbf{u}|| \to 0} g(\mathbf{u}^*, \delta\mathbf{u}) = 0.$$

Thus there exists a neighborhood of $\mathbf{u}^*$ with $||\delta\mathbf{u}|| < \epsilon$, where $||\delta\mathbf{u}|| \cdot g(\mathbf{u}^*, \delta\mathbf{u})$ is small enough so that δJ dominates the expression for ΔJ.
Now let us select the special variation $\delta\mathbf{u} = \alpha \cdot \delta\mathbf{u}^{(1)}$, where $\alpha > 0$ and $||\alpha \cdot \delta\mathbf{u}^{(1)}|| < \epsilon$. It is easy to see that $|| - \alpha \cdot \delta\mathbf{u}^{(1)}|| = |-1| \cdot ||\alpha \cdot \delta\mathbf{u}^{(1)}|| < \epsilon$. Hence

$$\begin{aligned}
& Sign\Big(\Delta J(\mathbf{u}^*, \alpha \cdot \delta\mathbf{u}^{(1)}) \cdot \Delta J(\mathbf{u}^*, -\alpha \cdot \delta\mathbf{u}^{(1)})\Big) \\
= \; & Sign\Big(\delta J(\mathbf{u}^*, \alpha \cdot \delta\mathbf{u}^{(1)}) \cdot \delta J(\mathbf{u}^*, -\alpha \cdot \delta\mathbf{u}^{(1)})\Big) \\
= \; & Sign\Big(\alpha \cdot (-\alpha) \cdot [\, \delta J(\mathbf{u}^*, \delta\mathbf{u}^{(1)}) \,]^2 \Big) \\
= \; & -1
\end{aligned}$$

We have shown that if $\delta J(\mathbf{u}^*, \delta\mathbf{u}^{(1)}) \neq 0$ then in an arbitrarily small neighborhood of $\mathbf{u}^*$ the increments $\Delta J(\mathbf{u}^*, \alpha \cdot \delta\mathbf{u}^{(1)})$ and $\Delta J(\mathbf{u}^*, -\alpha \cdot \delta\mathbf{u}^{(1)})$ have different signs and this contradicts the assumption that $\mathbf{u}^*$ is extremal. □

Let $\mathbf{u} = \mathbf{u}(t)$ be a function of t in the class Ω and $J(\mathbf{u})$ be a differentiable functional of $\mathbf{u}$. A nice way to find extremals for the functional J is the use of the so called Gâteaux-variation.

We define special variations $\delta\mathbf{u} = \delta\mathbf{u}(t) := \alpha\boldsymbol{v} = \alpha\boldsymbol{v}(t)$ with $\alpha \in \mathbb{R}$ and $\boldsymbol{v} : [t_0, t_f] \to \mathbb{R}^n$. Assume that $\mathbf{u}^* = \mathbf{u}^*(t)$ is an extremal of J and let

$$\mathbf{u}_\alpha(t) \quad = \quad \mathbf{u}^*(t) + \alpha\boldsymbol{v}(t)$$

such that $\mathbf{u}_\alpha \in \Omega$ for all $-\alpha_0 < \alpha < \alpha_0$. The **real** function

$$h(\alpha) \quad = \quad J\left(\mathbf{u}^*(t) + \alpha\boldsymbol{v}(t)\right) \quad = \quad J\left(\mathbf{u}_\alpha(t)\right)$$

has an extremal in $\alpha = 0$. If h is differentiable, then

$$0 \quad = \quad \frac{d}{d\alpha}h(\alpha)\bigg|_{\alpha=0} \quad = \quad \frac{d}{d\alpha}J\left(\mathbf{u}_\alpha(t)\right)\bigg|_{\alpha=0}.$$

Definition 1.5 *Let $J : \Omega \to \mathbb{R}$ be a (differentiable) functional and $\mathbf{u}, \mathbf{u} + \alpha\boldsymbol{v} \in \Omega$ for all $-\alpha_0 < \alpha < \alpha_0$. Then*

$$GJ(\mathbf{u}, \boldsymbol{v}) \quad = \quad \frac{d}{d\alpha}J\left(\mathbf{u} + \alpha\boldsymbol{v}\right)\bigg|_{\alpha=0}.$$

is called Gâteaux-variation of J in $\mathbf{u}$ and in direction $\boldsymbol{v}$. The 2. Gâteaux-variation of J in $\mathbf{u}$ and in direction $\boldsymbol{v}$ is defined by

$$G^{(2)}J(\mathbf{u}, \boldsymbol{v}) \quad = \quad \frac{d^2}{d\alpha^2}J\left(\mathbf{u} + \alpha\boldsymbol{v}\right)\bigg|_{\alpha=0}.$$

Theorem 1.2 (Fundamental Theorem of calculus of variations)

Assume that the functions in Ω are not constrained by any boundaries. If $\mathbf{u}^*$ is an extremal of the differentiable functional J, then the Gâteaux-variation of J must vanish on $\mathbf{u}^*$, that is for all $\boldsymbol{v}$ with $\mathbf{u}^* + \alpha\boldsymbol{v} \in \Omega$ for all $-\alpha_0 < \alpha < \alpha_0$ we have

$$\boxed{GJ(\mathbf{u}^*, \boldsymbol{v}) \; = \; 0}$$

Example 1.2 *Let $u = u(t)$ be a continuous scalar function defined for $t \in [0, 1]$ and let*

$$J(u) \quad := \quad \int_0^1 [u^2 + 2u] \; dt.$$

Then

$$\begin{aligned} GJ(u, \upsilon) &= \left. \frac{d}{d\alpha} J\left(u + \alpha\upsilon\right) \right|_{\alpha=0} \\ &= \int_0^1 \left. \frac{d}{d\alpha} [(u + \alpha\upsilon)^2 + 2(u + \alpha\upsilon)] \right|_{\alpha=0} \; dt \\ &= \int_0^1 \left. [2(u + \alpha\upsilon)\upsilon + 2\upsilon] \right|_{\alpha=0} \; dt \\ &= \int_0^1 [2u + 2]\upsilon \; dt. \end{aligned}$$

1.3 Functionals of a single function

1.3.1 t_0, t_f, u_0 and u_f fixed

- Admissible functions:

$$\Omega \;=\; \{\, u : [t_0, t_f] \to \mathbb{R} \mid u \in C^1[t_0, t_f], u(t_0) = u_0, u(t_f) = u_f \,\}$$

- Functional:

$$J(u) \;=\; \int_{t_0}^{t_f} g(t, u, \dot{u})\; dt$$

  g has continuous first and second order partial derivatives with respect to all of its variables.

- Problem: Find $u^* \in \Omega$ for which J has a relative extremum.

Theorem 1.3 (Euler equation) *A necessary condition for $u^\star$ to be an extremal is:*

$$\boxed{0 \;=\; g_u(t, u^\star, \dot{u}^\star) - \frac{d}{dt}\, g_{\dot{u}}(t, u^\star, \dot{u}^\star)}$$

This is a non linear, time varying, hard-to-solve, second order differential equation:

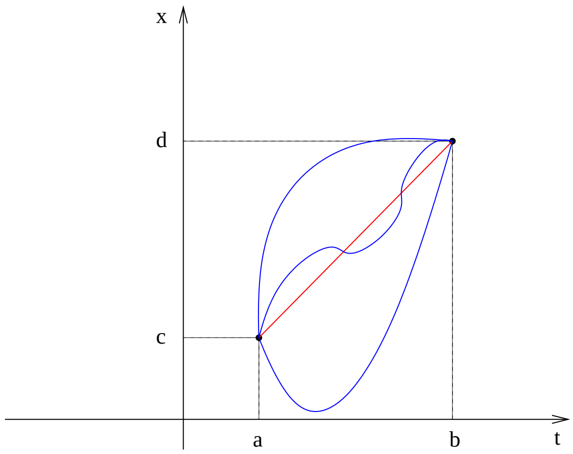
$$0 \;=\; g_u - \frac{d}{dt}\, g_{\dot{u}} \quad \to \quad 0 \;=\; g_u - g_{\dot{u}t} - g_{\dot{u}u}\dot{u} - g_{\dot{u}\dot{u}}\ddot{u}$$

Proof: For an extremal $u^\star$ we have for **all** υ with $u^* + \alpha\upsilon \in \Omega$ (this means $\upsilon(t_0) = \upsilon(t_f) = 0$) for all $-\alpha_0 < \alpha < \alpha_0$:

$$\begin{aligned}
0 \;=\; GJ(u^\star, \upsilon) \;&=\; \frac{d}{d\alpha}\left(\int_{t_0}^{t_f} g(t, u^\star + \alpha\upsilon, \dot{u}^\star + \alpha\dot{\upsilon})\; dt\right)\Bigg|_{\alpha=0} \\
&=\; \int_{t_0}^{t_f} \frac{d}{d\alpha}\, g(t, u^\star + \alpha\upsilon, \dot{u}^\star + \alpha\dot{\upsilon})\big|_{\alpha=0}\; dt \\
&=\; \int_{t_0}^{t_f} g_u\; \upsilon\; dt + \int_{t_0}^{t_f} \underbrace{g_{\dot{u}}}_{=f}\; \overbrace{\dot{\upsilon}}^{=\dot{h}}\; dt \\
&=\; \int_{t_0}^{t_f} g_u\; \upsilon\; dt + \underbrace{\Big[g_{\dot{u}}\upsilon\Big]_{t_0}^{t_f}}_{=0} - \int_{t_0}^{t_f} \frac{d}{dt}\, g_{\dot{u}}\; \upsilon\; dt \\
&=\; \int_{t_0}^{t_f} \left[\, g_u - \frac{d}{dt}\, g_{\dot{u}} \,\right] \upsilon\; dt.
\end{aligned}$$

We see: $g_u - \dfrac{d}{dt}\, g_{\dot{u}} \;=\; 0.$ $\square$

Example 1.3 *We are looking for the shortest path from (a, c) to (b, d).*



- $\Omega = \{ \; x : [a, b] \to \mathbb{R} \mid x \; C^2 \; function, x(a) = c \; and \; x(b) = d \; \}$
- *length of the graph:*

$$J(x) \; = \; \int_a^b \sqrt{1 + \dot{x}(t)^2} \, dt \; \text{and} \; g(t, x, \dot{x}) \; = \; \sqrt{1 + \dot{x}(t)^2}$$

- *Euler equation:*

$$g_x \; = \; 0 \qquad g_{\dot{x}} \; = \; \frac{\dot{x}}{\sqrt{1 + \dot{x}(t)^2}}$$

$$0 \; = \; 0 - \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1 + \dot{x}(t)^2}} \right)$$

- *Both the function between the brackets and $\dot{x} = \dot{x}(t)$ must be constant:*

$$\frac{\dot{x}}{\sqrt{1 + \dot{x}(t)^2}} \; = \; k \qquad \dot{x}(t) \; = \; \pm\sqrt{\frac{k^2}{1 - k^2}} \; =: \; m$$

hence: $x(t) = mt + n$.

- *adapt the constants:*

$$x(a) = c \;\; \leftrightarrow \;\; ma + n = c$$
$$x(b) = d \;\; \leftrightarrow \;\; mb + n = d$$

hence

$$m \; = \; \frac{d - c}{b - a} \quad \text{and} \quad n \; = \; \frac{cb - da}{b - a}$$

Example 1.4 (Ramsey-problem) *Consider an economy evolving over time where $K = K(t)$ denotes the capital stock, $C = C(t)$ the consumption and $Y = Y(t)$ the net national product at the time t. Suppose that $f(K) = Y$ where $f'(K) > 0$ (strictly increasing) and $f''(K) < 0$ (concave).*

- *For each time assume that*

$$f(K(t)) = C(t) + \dot{K}(t) \tag{1}$$

  which means that output $Y(t) = f(K(t))$ is divided between consumption $C(t)$ and investment $\dot{K}(t)$.

- *Let $K(0) = K_0$ be the given capital stock existing today at $t = 0$ and suppose that there is a fixed planning period $[0, T]$.*

- *For each choice of investment function $\dot{K}(t)$ on the interval $[0, T]$ the capital is determined by*

$$K(t) = K_0 + \int_0^t \dot{K}(\tau)\, d\tau \tag{2}$$

  and (1) determines $C(t)$.

- *Assume that the society has a utility function U, where $U = U(C)$ is the utility flow the country enjoys when the total consumption is C. Suppose $U'(C) > 0$ and $U''(C) < 0$.*

- *Introduce a discount rate r to reflect the idea that the present may matter more than the future. For each $t \geq 0$ multiply $U(C(t))$ by the discount factor e^{-rt}.*

The question of J.P. Ramsey (22 February 1903 - 19 January 1930, British philosopher, mathematician and economist, friend of L. Wittgenstein) is: How much investment would be desirable? (compare: Ramsey, F.P. (1928) "A mathematical theory of savings". Economic Journal, 38.)

Higher consumption today leads to a lower rate of investment. This in turn results in a lower capital stock and reduces the possibilities for future consumption. A way must be found to reconcile the conflict between higher consumption now and more investment in the future.

The goal of investment policy is to choose $K(t)$ for $t \in [0, T]$ in order to make the total discounted utility over the period as large as possible:

Find the function $K = K(t)$, with $K(0) = K_0$ (and $K(T) = K_T$) that maximizes

$$J(K) = \int_0^T U(\underbrace{f(K) - \dot{K}}_{=C(t)}) \cdot e^{-rt} dt \tag{3}$$

We see that

$$u(t, K, \dot{K}) = U(\overbrace{f(K) - \dot{K}}^{=C}) \cdot e^{-rt}$$

and

$$
\begin{aligned}
u_K(t, K, \dot{K}) &= U'(C) \cdot f'(K) \cdot e^{-rt} \\
u_{\dot{K}}(t, K, \dot{K}) &= U'(C) \cdot (-1) \cdot e^{-rt} \\
\frac{d}{dt} u_{\dot{K}}(t, K, \dot{K}) &= -U''(C) \cdot \dot{C} \cdot e^{-rt} + r \cdot U'(C) \cdot e^{-rt}
\end{aligned}
$$

The Euler equation is

$$
0 = U'(C) \cdot f'(K) \cdot e^{-rt} + U''(C) \cdot \dot{C} \cdot e^{-rt} - r \cdot U'(C) \cdot e^{-rt}
$$

or

$$
0 = U''(C) \cdot \dot{C} + U'(C) \cdot (f'(K) - r),
$$

resp. with $C = f(K) - \dot{K}$ *and* $\dot{C} = f'(K) \cdot \dot{K} - \ddot{K}$ *we get the differential equation for* K*:*

$$
0 = \ddot{K} - f'(K) \cdot \dot{K} - \frac{U'(C)}{U''(C)} (f'(K) - r).
$$

1.3.2 t_0, t_f, u_0 fixed and u_f free

- Admissible functions:

$$\Omega \;=\; \{\; u : [t_0, t_f] \to \mathbb{R} \mid u \in C^1[t_0, t_f], u(t_0) = u_0 \;\}$$

- Functional:

$$J(u) \;=\; \int_{t_0}^{t_f} g(t, u, \dot{u})\, dt$$

  g has continuous first and second order partial derivatives with respect to all of its variables.

- Problem: Find $u^* \in \Omega$ for which J has a relative extremum.

Theorem 1.4 (Euler equation and natural boundary condition)
Necessary conditions for $u^\star$ to be an extremal are:

$$0 \;=\; g_u(t, u^\star, \dot{u}^\star) - \frac{d}{dt}\, g_{\dot{u}}(t, u^\star, \dot{u}^\star) \quad \textit{and} \quad g_{\dot{u}}(t_f, u^\star(t_f), \dot{u}^\star(t_f)) \;=\; 0$$

Proof:
We have

$$GJ(u, v) \;=\; \Big[g_{\dot{u}}\; v \Big]_{t_0}^{t_f} + \int_{t_0}^{t_f} \left[g_u - \frac{d}{dt}\, g_{\dot{u}} \right] v\, dt \qquad (*)$$

and $v(t_0) = 0$ but $v(t_f)$ is arbitrary. For an extremal $u^\star$ we know $GJ(u^*, v) = 0$.

Now we show that the integral in $(*)$ must be zero in an extremal. Suppose that the curve u^* is an extremal for the free endpoint problem with $u^\star(t_f) = u_f$. Then the curve $u^\star$ must be an extremal for the fixed endpoint problem. Therefore $u^\star$ must be a solution of the Euler equation and the integral in $(*)$ must be zero in an extremal.

We see that $g_{\dot{u}}(t_f, u^\star(t_f), \dot{u}^\star(t_f))v(t_f) \;=\; 0$ but since $u(t_f)$ is free, $v(t_f)$ is arbitrary, therefore it is necessary that $g_{\dot{u}}(t_f, u^\star(t_f), \dot{u}^\star(t_f)) \;=\; 0$. □

1.4 Functionals involving several independent functions

1.4.1 t_0, t_f, $\mathbf{u}_0$ and $\mathbf{u}_f$ fixed

- Admissible functions:

$$\Omega \;=\; \{\; \mathbf{u}(t) = (u_1(t), \ldots, u_n(t)) : [t_0, t_f] \to \mathbb{R}^n \mid u_i \in C^1[t_0, t_f], \mathbf{u}(t_0) = \mathbf{u}_0, \mathbf{u}(t_f) = \mathbf{u}_f \;\}$$

- Functional:

$$J(\mathbf{u}) \;=\; \int_{t_0}^{t_f} g(t, \mathbf{u}, \dot{\mathbf{u}})\; dt$$

  g has continuous first and second order partial derivatives with respect to all of its variables.

- Problem: Find $\mathbf{u}^* \in \Omega$ for which J has a relative extremum.

Theorem 1.5 (Euler equation) *A necessary condition for $u^\star$ to be an extremal is:*

$$\boxed{0 \;=\; g_{\mathbf{u}}(t, \mathbf{u}^\star, \dot{\mathbf{u}}^\star) - \frac{d}{dt}\, g_{\dot{\mathbf{u}}}(t, \mathbf{u}^\star, \dot{\mathbf{u}}^\star)}$$

By using the chain rule and with

$$g_{\mathbf{u}} = \begin{pmatrix} g_{u_1} \\ \vdots \\ g_{u_n} \end{pmatrix} \quad g_{\dot{\mathbf{u}}t} = \begin{pmatrix} g_{\dot{u}_1 t} \\ \vdots \\ g_{\dot{u}_n t} \end{pmatrix} \quad g_{\dot{\mathbf{u}}\mathbf{u}} = \begin{pmatrix} g_{\dot{u}_1 u_1} & \cdots & g_{\dot{u}_1 u_n} \\ \vdots & \ddots & \vdots \\ g_{\dot{u}_n u_1} & \cdots & g_{\dot{u}_n u_n} \end{pmatrix} \quad g_{\dot{\mathbf{u}}\dot{\mathbf{u}}} = \begin{pmatrix} g_{\dot{u}_1 \dot{u}_1} & \cdots & g_{\dot{u}_1 \dot{u}_n} \\ \vdots & & \vdots \\ g_{\dot{u}_n \dot{u}_1} & \cdots & g_{\dot{u}_n \dot{u}_n} \end{pmatrix}$$

the Euler equation can be written as

$$\mathbf{0} \;=\; g_{\mathbf{u}} - \frac{d}{dt}\, g_{\dot{\mathbf{u}}} \quad \to \quad \mathbf{0} \;=\; g_{\mathbf{u}} - g_{\dot{\mathbf{u}}t} - g_{\dot{\mathbf{u}}\mathbf{u}}\dot{\mathbf{u}} - g_{\dot{\mathbf{u}}\dot{\mathbf{u}}}\ddot{\mathbf{u}}$$

2 Optimal control problems

2.1 Introduction and Examples

Optimal control theory is a modern extension of the classical calculus of variations.

Consider a system whose state at time t is characterized by a number $x = x(t)$. The process that causes $x(t)$ to change can be controlled, at least partially, by a controll function $u(t)$. We assume that the rate of change of $x(t)$ depends on $t, x(t)$ and $u(t)$. The state at some initial point t_0 is typically known, $x(t_0) = x_0$. Hence, the evolution of $x(t)$ is described by a (controlled) differential equation

$$\dot{x}(t) \;=\; a(t, x(t), u(t)), \qquad\qquad x(t_0) = x_0.$$

Suppose we choose some control function $u(t)$. Inserting this function in the differential equation gives a first-order differential equation for $x(t)$ and because the initial value is fixed, a unique solution is usually obtained. By choosing different control functions, the system can be steered along different paths. As usual in economic analysis, assume that it is possible to measure (by a so called functional) the benefits associated with each path. The fundamental problem that we study is:

Among all pairs $(x(t), u(t))$ that obey the differential equation with given initial value (and that satisfy given constraints), find one that maximizes (or minimizes) the benefits.

Example 2.1 *We start with an example to describe a typical optimal control problem and the way to solve it.*

We assume that the state of a system is given by (only) one function $x = x(t)$ on an interval $[t_0, t_f] = [0, 1]$ (the so called state function of the process) of time t. The set of all (possible) admissible state functions for the process is denoted by X.

Furthermore we assume that there is a set of so called (admissible) control functions U (without any boundary), such that if we choose one function $u = u(t) \in U$ then the evolution of the system can be described by the differential equation

$$\dot{x} \;=\; u \quad \textit{with the initial condition} \quad x(0) = x_0.$$

By choosing different control functions, the system can be steered along different trajectories.

Examples:

- *If $u = u(t) = 1$ then $x(t) \;=\; \int_0^t 1 d\tau \;=\; t + x_0$.*
- *If $u = u(t) = c$ then $x(t) \;=\; \int_0^t c d\tau \;=\; ct + x_0$.*
- *If $u = u(t) = t^2$ then $x(t) \;=\; \int_0^t \tau^2 d\tau \;=\; \frac{1}{3}t^3 + x_0$.*

The last important part of an optimal control problem is the so called performance measure, which assigns a unique real number to each possible trajectory of the system

$$J(u) \;=\; \int_0^1 (u^2 + tx - 1)dt$$

and the optimal control problem is: **Find an admissible control function u^* which causes the system to follow an admissible state function x^* that minimizes the performance measure.**

Solution:

- *We define the so called augmented functional:*

$$J_a(u) \;:=\; \int_0^1 \left[(u^2 + tx - 1) + \lambda(u - \dot{x})\right]\; dt$$

  We see that $J_a(u) \;=\; J(u)$ for **all** *functions $\lambda = \lambda(t)$, if $\dot{x} = u$.*

- *Integration by parts:*

$$\begin{aligned} J_a(u) &= \int_0^1 \left[u^2 + tx - 1 + \lambda u - \lambda\dot{x}\right]\; dt \\ &= \int_0^1 \left[u^2 + tx - 1 + \lambda u\right]\; dt - \int_0^1 \lambda\dot{x}\; dt \\ &= \int_0^1 \left[u^2 + tx - 1 + \lambda u\right]\; dt - \lambda x\big|_0^1 + \int_0^1 \dot{\lambda} x\; dt \\ &= -\lambda(1)x(1) + \lambda(0)x(0) + \int_0^1 \left[u^2 + tx - 1 + \lambda u + \dot{\lambda} x\right]\; dt \end{aligned}$$

- *Now let u^* (and x^*) be the solution of this optimal control problem. Let*

$$u_\epsilon(t) \;:=\; u^*(t) + \epsilon \cdot \delta(t)$$

  for all real numbers $-\epsilon_0 < \epsilon < \epsilon_0$ be a admissible control function, called a variation of the solution u^*. The associated state function is given by

$$x_\epsilon(t) \;=\; \int_0^t u_\epsilon(\tau)d\tau \;=\; x^*(t) + \epsilon \cdot \Delta(t)$$

  with $\Delta(0) = 0$ because we would like to have $x_\epsilon(0) = x_0$ for all admissible state functions.

- *Define the* **real** *function*

$$h(\epsilon) \;:=\; J_a(u_\epsilon(t)) \;=\; J_a(u^*(t) + \epsilon \cdot \delta(t))$$

  and we should have that

$$\frac{\partial}{\partial \epsilon} h(\epsilon)\bigg|_{\epsilon=0} \;=\; 0.$$

- *Concrete:*

$$\begin{aligned} h(\epsilon) &= -\lambda(1)(x^*(1) + \epsilon \cdot \Delta(1)) + \lambda(0)(x^*(0) + \epsilon \cdot \Delta(0)) \\ &\quad + \int_0^1 \Big[(u^* + \epsilon \cdot \delta)^2 + t(x^* + \epsilon \cdot \Delta) - 1 + \lambda(u^* + \epsilon \cdot \delta) + \dot{\lambda}(x^* + \epsilon \cdot \Delta)\Big] \, dt \end{aligned}$$

$$\begin{aligned} 0 = \frac{\partial}{\partial \epsilon} h(\epsilon)\Big|_{\epsilon=0} &= -\lambda(1)\Delta(1) + \lambda(0)\Delta(0) + \int_0^1 \Big[2u^*\delta + t\Delta + \lambda\delta + \dot{\lambda}\Delta\Big] \, dt \\ &= -\lambda(1)\Delta(1) + \lambda(0) \underbrace{\Delta(0)}_{=0} + \int_0^1 \Big[(2u^* + \lambda)\delta + (t + \dot{\lambda})\Delta\Big] \, dt \end{aligned}$$

for **all** *suitable functions* δ *(and* Δ*).*

- *We have to choose:*

$$\begin{aligned} 0 = t + \dot{\lambda} &\longrightarrow \lambda^*(t) = -\frac{1}{2}t^2 + c \\ 0 = \lambda^*(1) &\longrightarrow \lambda^*(t) = -\frac{1}{2}t^2 + \frac{1}{2} \\ 0 = 2u^* + \lambda &\longrightarrow u^*(t) = \frac{1}{4}t^2 - \frac{1}{4} \\ \dot{x}^* = u^* &\longrightarrow x^*(t) = \frac{1}{12}t^3 - \frac{1}{4}t + x_0 \end{aligned}$$

Example 2.2 (Oil extraction) *We denote by $x(t) = x_1(t)$ the amount of oil in a reservoir at time t and we assume that $x(0) = K$ barrels of oil. Then $u(t) = u_1(t)$ is the rate of extraction, which means $\dot{x}(t) = -u(t)$ and by integration of this equation we get*

$$\int_0^t \dot{x}(\tau)\, d\tau \;=\; -\int_0^t u(\tau)\, d\tau \qquad \text{or} \qquad x(t) \;=\; K - \int_0^t u(\tau)\, d\tau$$

The amount of oil left at time t is equal to the initial amount K, minus the total amount that has been extracted during the time span $[0, t]$.

- *state and control variables: $x(t)$ and $u(t)$*
- *state equation: $\dot{x}(t) = -u(t)$*
- *state constraints: $x(0) = K$*
- *control constraints: $0 \leq u(t) \leq M$*
- *performance measure:*

  Suppose that the market price of oil at time t is known to be $q(t)$, so that the sales revenue per unit of time t is $q(t)u(t)$.

  Assume further, that the cost $C = C(t, x(t), u(t))$ per unit of time depends on $t, x = x(t)$ and $u = u(t)$. The instantaneous profit per unit of time is

  $$\pi(t, x(t), u(t)) \;=\; q(t)u(t) - C(t, x(t), u(t)).$$

  If the discount rate is r, the total discounted profit over the interval $[t_0 = 0, t_f]$ is a useful performance measure:

  $$J \;=\; \int_0^{t_f} \Big[\, q(t)u(t) - C(t, x(t), u(t)) \,\Big]\, e^{-rt}\, dt$$

  There are at least two standard problems:

  1. *Find an admissible $u(t)$ that maximizes the performance measure J over a fixed extraction period $[0, t_f]$.*
  2. *Find an admissible $u(t)$ and also an optimal terminal time t_f that maximizes the performance measure J.*

Example 2.3 (The car) *A car is to be driven in a straight line and let $d(t)$ be the distance from 0. The car can be accelerated by using the throttle or decelerated by using the brake. Then we have Newton's Law:*

$$\ddot{d}(t) = \alpha(t) + \beta(t)$$

where $\alpha(t) \geq 0$ is the throttle acceleration and $\beta(t) \leq 0$ is the braking acceleration.

- *state and control variables*

$$\begin{aligned} x_1(t) &= d(t) & \qquad u_1(t) &= \alpha(t) \\ x_2(t) &= \dot{d}(t) & \qquad u_2(t) &= \beta(t) \end{aligned}$$

- *state equations*

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u_1(t) + u_2(t) \end{aligned}$$

or

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{u}(t)$$

- *state constraints*

$$\begin{aligned} x_1(t_0) &= 0 & \qquad 0 \leq\ & x_1(t) \leq e \\ x_1(t_f) &= e & \qquad 0 \leq\ & x_2(t) \\ x_2(t_0) = x_2(t_f) &= 0 & & \end{aligned}$$

- *control constraints*

  We know that the acceleration is bounded by some upper limit which depends on the capability of the engine and that the maximum deceleration is limited by the braking system parameters.

$$0 \leq u_1(t) \leq M_1 \qquad -M_2 \leq u_2(t) \leq 0$$

  In addition, if the car starts with G gallons of gas and there are no service stations on the way, another constraint is

$$\int_{t_0}^{t_f} [k_1 u_1(t) + k_2 x_2(t)]\, dt \leq G$$

  which assumes that the rate of gas consumption is proportional to both acceleration and speed (with constants of proportionality k_1 and k_2).

- *performance measure*

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt$$

2.2 The mathematical model

A nontrivial part of any control problem is modeling the process. The objective is to obtain the simplest mathematical description that adequately predicts the response of the physical system to all anticipated inputs.

Our discussion will be restricted to systems described by **ordinary differential equations** (in state variable form).

State function, control function and state equation

Definition 2.1 *If*

$$x_1(t), x_2(t), \ldots, x_n(t) \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

are the state variables and the state vector or state function of a process at time t,

$$u_1(t), u_2(t), \ldots, u_m(t) \quad \text{and} \quad \mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{pmatrix}$$

are control inputs and the control vector or control function to the process at time t, then the process (or system) may be described by n first-order differential equations

$$\begin{aligned} \dot{x}_1(t) &= a_1(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), t) \\ \dot{x}_2(t) &= a_2(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), t) \\ \vdots \quad &\quad \vdots \\ \dot{x}_n(t) &= a_n(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), t) \end{aligned}$$

or shortly by the state equation

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) \qquad (*)$$

Suppose we choose some control function $\mathbf{u}(t)$ and an initial point $\mathbf{x}(t_0) = \mathbf{x}_0$. Inserting this function into $(*)$ gives a first-order system of ordinary differential equations for $\mathbf{x}(t)$. Because the initial point is fixed, a unique solution of $(*)$ is usually obtained. By choosing different control functions $\mathbf{u}(t)$, the system can be steered along different paths, not all of which are equally desirable.

Systems are described by the terms linear, nonlinear, stationary (or time-invariant) and time-varying according to the form of the state equation.

- non-linear and time-varying (general form)

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)$$

- linear and time-varying

$$\dot{\mathbf{x}}(t) \;=\; \mathbf{A(t)x(t)} + \mathbf{B(t)u(t)}$$

where $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ matrices with time varying entries,

- linear and time-invariant

$$\dot{\mathbf{x}}(t) \;=\; \mathbf{Ax(t)} + \mathbf{Bu(t)}$$

where A and B are constant matrices.

The physical quantities that can be measured are called the outputs and are denoted by $\mathbf{y}(t) = (y_1(t), \ldots, y_q(t))$. If the outputs are nonlinear, time-varying functions of the states and controls, we write the outputequations $\mathbf{y}(t) = \mathbf{c}(t, \mathbf{x}(t), \mathbf{u}(t))$.

2.3 Physical constraints

Definition 2.2 *A control function which satisfies the control constraints during the entire time interval $[t_0, t_f]$ is called an admissible control function. Let U denote the set of all admissible control functions.*

A state function which satisfies the state function constraints during the entire time interval is called an admissible state function. Let X denote the set of all admissible state functions.

In general, the final state of a system will be required to lie in a specified region S, the so-called target set, of the $(n+1)$-dimensional state-time space. If the final state and the final time are fixed, then S is a point.

2.4 The performance measure

2.4.1 Introduction

In order to evaluate the performance of a system, the designer has to select a performance measure. An optimal control function is defined as one that minimizes (or maximizes) the performance measure.

In all that follows it will be assumed that the performance of a system is evaluated by a measure of the form

$$\boxed{J \;=\; J(\mathbf{u}(t)) \;=\; h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt}$$

where

- t_0, t_f are the initial and the final time,
- h and g are scalar functions and

- t_f may be specified or free.

Starting from the initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ and applying a control signal $\mathbf{u}(t)$ for $t \in [t_0, t_f]$, causes the system to follow some state trajectory. The performance measure assigns a unique real number to each trajectory of the system.

2.4.2 Typical performance measures

Let us discuss some typical control problems to provide some motivation for the selection of a performance measure.

Minimum-time problems

To transfer a system from an arbitrary initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ to a specified target set S in minimum time, the perrormance measure to be minimized is

$$J \;=\; t_f - t_0 \;=\; \int_{t_0}^{t_f} dt$$

with t_f the first instant of time when $\mathbf{x}(t)$ and S intersect.

Terminal control problems

To minimize the deviation of the final state of a system from its desired value $\mathbf{r}(t_f)$, a possible perrormance measure to be minimized is

$$\begin{aligned} J &= \sum_{i=1}^{n} [\, x_i(t_f) - r_i(t_f) \,]^2 \\ &= [\, \mathbf{x}(t_f) - \mathbf{r}(t_f) \,]^T [\, \mathbf{x}(t_f) - \mathbf{r}(t_f) \,] \\ &= ||\, \mathbf{x}(t_f) - \mathbf{r}(t_f) \,||^2. \end{aligned}$$

To allow greater generallity, we can insert a real symmetric positive semi-definit $(n \times n)$ weighting matrix H to obtain

$$J \;=\; [\, \mathbf{x}(t_f) - \mathbf{r}(t_f) \,]^T \; H \; [\, \mathbf{x}(t_f) - \mathbf{r}(t_f) \,] \;=:\; ||\, \mathbf{x}(t_f) - \mathbf{r}(t_f) \,||_H^2$$

Tracking problems

To maintain the system state $\mathbf{x}(t)$ as close as possible to the desired state $\mathbf{r}(t)$ in the interval $[t_0, t_f]$, a possible perrormance measure to be minimized is

$$J \;=\; \int_{t_0}^{t_f} ||\, \mathbf{x}(t) - \mathbf{r}(t) \,||_{Q(t)}^2 \, dt$$

where $Q(t)$ is a a real symmetric positive semi-definit $(n \times n)$ matrix for all $t \in [t_0, t_f]$.

Minimal control effort problems

To transfer a system from an arbitrary initil state $\mathbf{x}(t_0) = \mathbf{x}_0$ to a specified target set S, with a minimum expenditure of control effort.The meaning of the term ,,control effort,, depends upon the particular application; therefore, the performance measure may assume various forms.

2.5 The optimal control problem

Definition 2.3 (The optimal control problem) *Find an admissible control function* $\mathbf{u}^*(t)$ *which causes the system* $\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)$ *to follow an admissible state trajectory (function)* $\mathbf{x}^*(t)$ *that minimizes the performance measure*

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt.$$

When we say that $\mathbf{u}^*(t)$ causes the performance measure to be minimized, we mean that

$$\begin{aligned} J^* &= h(\mathbf{x}^*(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\ dt \\ &\leq h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt \end{aligned}$$

for all $\mathbf{u} \in U$ which make $\mathbf{x} \in X$.

2.6 Pontryagin's Minimimum Principle

Notation: ∇ is the so called gradient.

$$
\begin{aligned}
\nabla_{\boldsymbol{x}} H &= \begin{pmatrix} H_{x_1} \\ H_{x_2} \\ \vdots \\ H_{x_n} \end{pmatrix} = \begin{pmatrix} \partial H/\partial x_1 \\ \partial H/\partial x_2 \\ \vdots \\ \partial H/\partial x_n \end{pmatrix} \\
\nabla_{\boldsymbol{u}} H &= \begin{pmatrix} H_{u_1} \\ H_{u_2} \\ \vdots \\ H_{u_m} \end{pmatrix} \\
\nabla_{\boldsymbol{\lambda}} H &= \begin{pmatrix} H_{\lambda_1} \\ H_{\lambda_2} \\ \vdots \\ H_{\lambda_n} \end{pmatrix}
\end{aligned}
$$

Theorem 2.1 *Let* $\mathbf{u}^*$ *(or* $(\mathbf{u}^*, \mathbf{x}^*)$*) be a solution of the problem: Minimize the performance measure*

$$
J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt
$$

for the system $\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)$.
We introduce Lagrange-multiliers $\boldsymbol{\lambda}(t) = (\lambda_1(t), \ldots, \lambda_n(t))$ *and define the Hamilton function*

$$
H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t) = g(\mathbf{x}, \mathbf{u}, t) + \boldsymbol{\lambda}^T \cdot \mathbf{a}(\mathbf{x}, \mathbf{u}, t).
$$

Necessary conditions for optimality: There exists a function $\boldsymbol{\lambda}^*$ *such that*

$$
\boxed{
\begin{array}{rcll}
\dot{\mathbf{x}}^* &=& \nabla_{\boldsymbol{\lambda}} H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, t) & \\
\dot{\boldsymbol{\lambda}}^* &=& -\nabla_{\boldsymbol{x}} H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, t) & \\
H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, t) &\leq& H(\mathbf{x}^*, \mathbf{u}\ , \boldsymbol{\lambda}^*, t) & \text{for all admissible } \mathbf{u}
\end{array}
}
$$

for all $t \in [t_0, t_f]$
If the set of all admissible $\mathbf{u}$ *has no boundary and the function* H *is convex in* $\mathbf{u}$*, then the conditions*

$$
\boxed{
\begin{array}{rcl}
\dot{\mathbf{x}}^* &=& \nabla_{\boldsymbol{\lambda}} H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, t) \\
\dot{\boldsymbol{\lambda}}^* &=& -\nabla_{\boldsymbol{x}} H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, t) \\
\mathbf{0} &=& \nabla_{\boldsymbol{u}} H(\mathbf{x}^*, \mathbf{u}\ , \boldsymbol{\lambda}^*, t)
\end{array}
}
$$

for all $t \in [t_0, t_f]$ *are necessary for minimizing.*