Discrete processes and difference equations

Keywords: differential and difference equations, linear difference equations (with constant coefficients), stability, systems of difference equations, linear systems of difference equations

Contents

1 Introduction

Let $x = x(t)$ be a quantity as a function of t. Suppose that for any Δt:

$$\Delta x \;=\; x(t+\Delta t) - x(t) \;=\; k \cdot x(t) \cdot \Delta t$$

If this equation is (approximately) correct for

- very small Δt ($\Delta t \to 0$), then we get a differential equation

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} \;=\; k \cdot x(t) \quad\longrightarrow\quad \dot{x} \;=\; k \cdot x$$

- $\Delta t \approx 1$, then we get a difference equation (with $x_t = x(t)$)

$$x(t+1) - x(t) \;=\; k \cdot x(t) \quad\longrightarrow\quad x_{t+1} \;=\; k \cdot x_t + x_t \;=\; (k+1) \cdot x_t$$

Difference equations are the discrete time counterparts of differential equations in continuous time.

Definition 1.1 *Let f be a function defined for all values of the variables. The general nth-order difference equation is*

$$x_{t+n} \;=\; f(t, x_t, x_{t+1}, \ldots, x_{t+n-1}) \quad t = 0, 1, 2, \ldots \qquad (\star)$$

If we require that $x_0, x_1, \ldots, x_{n-1}$ have given values (initial condition), then

$$\begin{aligned} x_n &= f(0, x_0, x_1, \ldots, x_{n-1}) \\ x_{n+1} &= f(1, x_1, x_2, \ldots, x_n) \;=\; f(1, x_1, x_2, \ldots, f(0, x_0, x_1, \ldots, x_{n-1})) \\ \ldots &\quad \ldots \end{aligned}$$

The solution of the difference equation $(\star)$ for all $t \geq n$ is uniquely determined (if it exists).

Definition 1.2 *The general solution of $(\star)$ is a function*

$$x_t \;=\; g(t; C_1, C_2, \ldots, C_n)$$

that depends on n arbitrary constants $C_1, C_2, \ldots, C_n$, satisfies $(\star)$ and has the property that every solution of $(\star)$ can be obtained by giving $C_1, C_2, \ldots, C_n$ appropriate values.

Example 1.1 *The general solution of $x_{t+2} \;=\; 5x_{t+1} - 6x_t + 4^t + t^2 + 3$ is*

$$x_t \;=\; A\, 2^t + B\, 3^t + \frac{1}{2}4^t + \frac{1}{2}t^2 + \frac{3}{2}t + 4$$

2 Linear difference equations

Theorem 2.1 *The general solution of the homogeneous linear difference equation*

$$x_{t+n} + a_1(t)\, x_{t+n-1} + \cdots + a_{n-1}(t)\, x_{t+1} + a_n(t)\, x_t \;=\; 0$$

with $a_n(t) \neq 0$ *is given by*

$$x_t \;=\; C_1\, u_t^{(1)} + \cdots + C_n\, u_t^{(n)}$$

where $u_t^{(1)}, \ldots, u_t^{(n)}$ *are* n *linearly independent solutions of the equation and* $C_1, C_2, \ldots, C_n$ *are abitrary constants.*

If $u_t^{(1)}, \ldots, u_t^{(n)}$ are solutions of the homogeneous linear difference equation, then

$$u_t^{(1)}, \ldots, u_t^{(n)} \text{ are lin. indep.} \quad \longleftrightarrow \quad \begin{vmatrix} u_0^{(1)} & \ldots & u_0^{(n)} \\ u_1^{(1)} & \ldots & u_1^{(n)} \\ \vdots & \vdots & \vdots \\ u_{n-1}^{(1)} & \ldots & u_{n-1}^{(n)} \end{vmatrix} \neq 0$$

Explanation:
If the determinant is zero, then the columns are linearly dependent and since $u_t^{(1)}, \ldots, u_t^{(n)}$ are solutions of the homogeneous linear difference equation, this dependence will propagate to $u_t^{(1)}, \ldots, u_t^{(n)}$ for **all** t.

Theorem 2.2 *The general solution of the nonhomogeneous linear difference equation*

$$x_{t+n} + a_1(t)\, x_{t+n-1} + \cdots + a_{n-1}(t)\, x_{t+1} + a_n(t)\, x_t \;=\; b_t$$

with $a_n(t) \neq 0$ *is given by*

$$x_t \;=\; C_1\, u_t^{(1)} + \cdots + C_n\, u_t^{(n)} + u_t^{\star}$$

where $C_1\, u_t^{(1)} + \cdots + C_n\, u_t^{(n)}$ *is the general solution of the corresponding homogeneous equation and* $u_t^{\star}$ *is a particular solution of the nonhomogeneous equation.*

3 Linear difference equations with constant coefficients

3.1 General facts

The general linear difference equation of nth order with constant coefficients takes the form

$$x_{t+n} + a_1\, x_{t+n-1} + \cdots + a_{n-1}\, x_{t+1} + a_n\, x_t \;=\; b_t \tag{1}$$

The corresponding homogeneous equation is

$$x_{t+n} + a_1\, x_{t+n-1} + \cdots + a_{n-1}\, x_{t+1} + a_n\, x_t \;=\; 0 \tag{2}$$

We try to find a solution of the homogeneous equation of the form $x_t = m^t$. Inserting this solution and cancelling the common factor m^t yields the characteristic equation

$$m^n + a_1\, m^{n-1} + \cdots + a_{n-1}\, m + a_n \;=\; 0. \tag{3}$$

This polynomial has exactly n (complex) solutions (counted with multiplicity).

Suppose that equation (3) has exactly n different real solutions $m_1, m_2, \ldots, m_n$. Then $m_1^t, m_2^t, \ldots, m_n^t$ all satisfy equation (2) and these functions are linearly independent. The general solution is

$$x_t \;=\; C_1\, m_1^t + \cdots + C_n\, m_n^t$$

This is **not** the general solution of (2) if (3) has multiple and /or complex solutions.

General method of solving:

Find the solutions of (3) together with their multiplicity.

- A real solution m_i with multiplicity 1 gives the one solution m_i^t.
- A real solution m_j with multiplicity $p > 1$ gives the p solutions $m_j^t, tm_j^t, \ldots, t^{p-1}m_j^t$.
- A pair of complex roots $\alpha \pm i\beta$, each with multiplicity 1, gives the two solutions $r^t \cos\theta t$ and $r^t \sin\theta t$, where $r = \sqrt{\alpha^2+\beta^2}$ and $\theta \in [0,\pi]$ satisfies $\cos\theta = \alpha/r$, $\sin\theta = \beta/r$.
- A pair of complex roots $\alpha \pm i\beta$, each with multiplicity $q > 1$, gives the $2q$ solutions $u, v, tu, tv, \ldots, t^{q-1}u, t^{q-1}v$, with $u = r^t \cos\theta t$ and $v = r^t \sin\theta t$, where $r = \sqrt{\alpha^2+\beta^2}$ and $\theta \in [0,\pi]$ satisfies $\cos\theta = \alpha/r$, $\sin\theta = \beta/r$.

In order to find the general solution of the nonhomogeneous equation (1), it remains to find a particular solution $u^\star$.

b_t	$u^\star$
$a_0 + a_1 t + \cdots + a_l t^l$	$A_0 + A_1 t + \cdots + A_l t^l$
ce^{ut}	Ce^{ut}
$c_1 \sin vt + c_2 \cos vt$	$C_1 \sin vt + C_2 \cos vt$

3.2 Stability

Definition 3.1 *The equation*

$$x_{t+n} + a_1\, x_{t+n-1} + \cdots + a_{n-1}\, x_{t+1} + a_n\, x_t \;=\; b_t$$

is called <u>*globally asymptotically stable*</u>*, if the general solution* $x_t = C_1\, u_t^{(1)} + \cdots + C_n\, u_t^{(n)}$ *of the associated homogeneous equation tends to* 0 *as* $t \to \infty$ *(for all values of the constants* $C_1, \ldots, C_n$*).*

Then the effect of the initial conditions "dies out" as $t \to \infty$.

Theorem 3.1 *The equation is globally asymptotically stable if and only if all roots of the characteristic polynomial of this equation have moduli strictly less than* 1*.*

Example 3.1 *Examine the stability of the difference equation* $x_{t+2} - \frac{1}{3}x_t \;=\; \sin t$*.*

Solution:

- *characteristic equation:* $m^2 - \frac{1}{3} \;=\; 0$*; solutions:* $m \;=\; \pm\sqrt{\frac{1}{3}}$
- *general solution of the associated homogeneous difference equation:*
  $x_t \;=\; C_1 \left(\sqrt{\frac{1}{3}}\right)^t + C_2 \left(-\sqrt{\frac{1}{3}}\right)^t$
- $\lim\limits_{t\to\infty} C_1 \left(\sqrt{\frac{1}{3}}\right)^t + C_2 \left(-\sqrt{\frac{1}{3}}\right)^t \;=\; 0$
- *equation is globally asymptotically stable*

3.3 First-order

3.3.1 The general solution

We consider the linear first-order difference equation with constant coefficients $x_{t+1}+ax_t = b$ where a, b, x_0 are real numbers.

- General solution of the homogeneous equation $x_{t+1} + ax_t \; = \; 0$:

  With $x_t^h = m^t$ we get the characteristic equation

$$m + a \; = \; 0 \quad \to \quad m \; = \; -a \quad \to \quad x_t^h \; = \; Cm^t \; = \; C(-a)^t$$

- Particular solution of the nonhomogeneous equation $x_{t+1} + ax_t \; = \; b$:

  With $x_t^\star = c$ we get

$$c + ac \; = \; b \quad \to \quad c \; = \; \frac{b}{1+a} \quad \text{if} \;\; a \neq -1$$

  If $a = -1$ a particular solution of $x_{t+1} - x_t \; = \; b$ is $x_t^\star = bt$.

- The general solution of $x_{t+1} + ax_t = b$ with $a \neq -1$ is

$$x_t \;\; = \;\; C\,(-a)^t + \frac{b}{1+a} \;\; = \;\; \left(x_0 - \frac{b}{1+a} \right)\,(-a)^t + \frac{b}{1+a}$$

  The general solution of $x_{t+1} - x_t = b$ is

$$x_t \;\; = \;\; C\,(-a)^t + bt \;\; = \;\; x_0\,(-a)^t + bt$$

3.3.2 Economic application: Cobweb models

Let D_t be the demand, S_t the supply and p_t the price of a good at time $t = 0, 1, 2, \ldots$ We use a linear model $(a, b, c, d > 0)$ and assume that

- $D_t = f(p_t) \;=\; -ap_t + b$: The demand is a function of the price in the same period.
- $S_{t+1} = g(p_t) \;=\; cp_t - d$: The supply is a function of the price in the preceding period.
- price formation: $S_{t+1} = D_{t+1}$ or $g(p_t) = f(p_{t+1})$ or

$$-ap_{t+1} + b \;=\; cp_t - d \qquad \to \qquad p_{t+1} \;=\; -\frac{c}{a}\,p_t + \frac{b+d}{a}$$

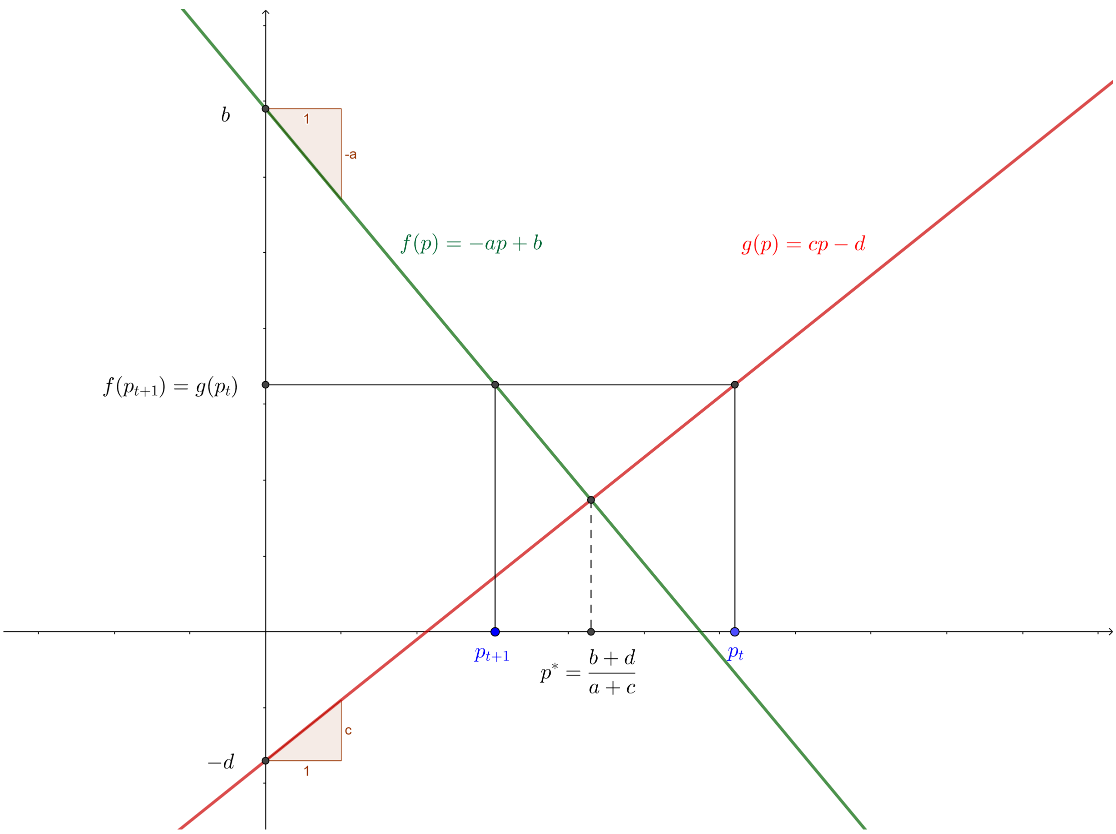
The solution of this first order linear difference equation is

$$p_t \;=\; \left(p_0 - \frac{b+d}{a+c}\right)\left(-\frac{c}{a}\right)^t + \frac{b+d}{a+c}$$

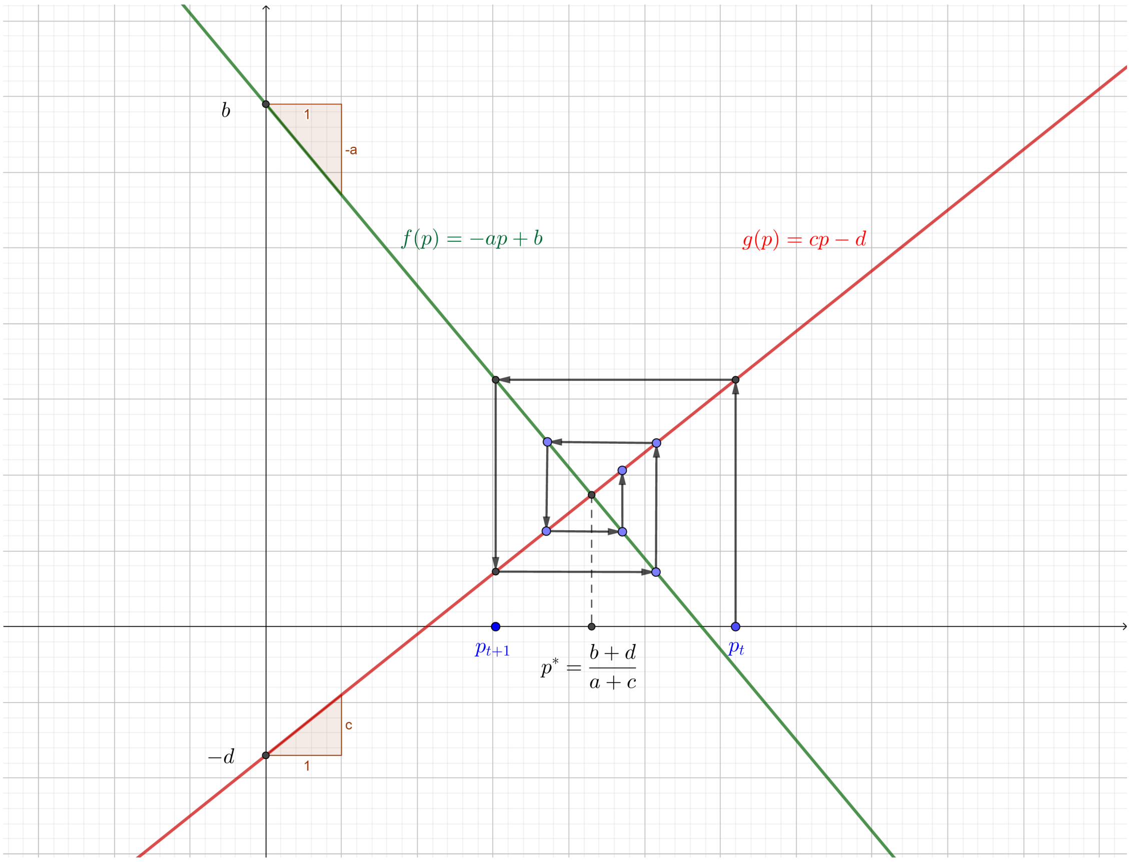
The number

$$p^* \;=\; \frac{b+d}{a+c}$$

is the long-term balance price (we have $g(p^*) = f(p^*)$).



Essential properties of the solution depends only on the term $-\frac{c}{a}$.



Example 3.2 *Investigate the solution* p_t *if*

1. $\left|-\dfrac{c}{a}\right| < 1,$
2. $\left|-\dfrac{c}{a}\right| = 1$ *and*
3. $\left|-\dfrac{c}{a}\right| > 1.$

3.4 Second-order and homogeneous

We consider the linear second-order homogeneous difference equation $x_{t+2}+ax_{t+1}+bx_t = 0$ where a, b, x_0, x_1 are real numbers. With $x_t = m^t$ we get the characteristic equation

$$m^2 + a\,m + b = \;0 \;\;\rightarrow\;\; m_{1,2} \;=\; -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}$$

Theorem 3.2 *The general solution of* $x_{t+2}+ax_{t+1}+bx_t = 0$ *depends on the characteristic equation as follows:*

1. *If* $\frac{a^2}{4} - b > 0$ *then* $x_t = C_1 m_1^t + C_2 m_2^t$.
2. *If* $\frac{a^2}{4} - b = 0$ *then* $x_t = (C_1 + C_2 t)m^t$ *with* $m = -\frac{a}{2}$.
3. *If* $\frac{a^2}{4} - b < 0$ *then* $x_t = r^t(C_1 \cos\beta t + C_2 \sin\beta t)$ *with* $r = \sqrt{b}$ *and* $\cos\beta = -\frac{a}{2\sqrt{b}}$ *and* $\beta \in [0, \pi]$

Example 3.3 *Difference equation:* $x_{t+2} - 5x_{t+1} + 6x_t = 0$
Characteristic equation: $m^2 - 5m + 6 = 0$ *and* $m_1 = 2,\ m_2 = 3$
General solution: $x_t = C_1 2^t + C_2 3^t$

Example 3.4 *Difference equation:* $x_{t+2} - 6x_{t+1} + x_t = 0$
Characteristic equation: $m^2 - 6m + 1 = 0$ *and* $m_1 = m_2 = 3$
General solution: $x_t = (C_1 + C_2 t)3^t$

Example 3.5 *Difference equation:* $x_{t+2} - x_{t+1} + x_t = 0$
Characteristic equation: $m^2 - m + 1 = 0$ *and* $r = \sqrt{b} = 1,\ \cos\beta = 1/2$ *so* $\beta = \frac{\pi}{3}$
General solution: $x_t = C_1 \cos\frac{\pi}{3}t + C_2 \sin\frac{\pi}{3}t$

4 Systems of difference equations

4.1 Introduction

Definition 4.1 *A system of first order difference equations in the n unknown functions $x_{1,t}, x_{2,t}, \ldots, x_{n,t}$ can be expressed in the so-called normal form*

$$\begin{aligned} x_{1,t+1} &= f_1(t, x_{1,t}, \ldots, x_{n,t}) \\ \vdots \quad & \quad \vdots \\ x_{n,t+1} &= f_n(t, x_{1,t}, \ldots, x_{n,t}) \end{aligned} \qquad (*)$$

If $x_{1,0}, x_{2,0}, \ldots, x_{n,0}$ are specified, then $x_{1,1}, x_{2,1}, \ldots, x_{n,1}$ are found by substituting $t = 0$ in the system, next $x_{1,2}, x_{2,2}, \ldots, x_{n,2}$ are found by substituting $t = 1$ etc.

Theorem 4.1 *The solution of the system is uniquely determined by the values of $x_{1,0}, x_{2,0}, \ldots, x_{n,0}$.*

Definition 4.2 *The general solution of the system $(*)$ is given by n functions*

$$\begin{aligned} x_{1,t} &= g_1(t, C_1, \ldots, C_n) \\ \vdots \quad & \quad \vdots \\ x_{n,t} &= g_n(t, C_1, \ldots, C_n) \end{aligned} \qquad (**)$$

with the property that an arbitrary solution is obtained from $(**)$ by giving $C_1, \ldots, C_n$ appropriate values.

Example 4.1 *Find the general solution of the system*

$$\begin{aligned} x_{t+1} &= \frac{1}{2}x_t + \frac{1}{3}y_t \quad (I) \\ y_{t+1} &= \frac{1}{2}x_t + \frac{2}{3}y_t \quad (II) \end{aligned}$$

Solution:

- (I) $x_{t+1} = \frac{1}{2}x_t + \frac{1}{3}y_t \leftrightarrow y_t = 3x_{t+1} - \frac{3}{2}x_t$ and $y_{t+1} = 3x_{t+2} - \frac{3}{2}x_{t+1}$
- in (II) $y_{t+1} = \frac{1}{2}x_t + \frac{2}{3}y_t$ we get:

  $$3x_{t+2} - \frac{3}{2}x_{t+1} = \frac{1}{2}x_t + \frac{2}{3}\left(3x_{t+1} - \frac{3}{2}x_t\right) \leftrightarrow x_{t+2} - \frac{7}{6}x_{t+1} + \frac{1}{6}x_t = 0$$

  a linear homogeneous difference equation of order 2 with
- general solution: $x_t = C_1 + C_2\left(\frac{1}{6}\right)^t$
- $y_t = 3x_{t+1} - \frac{3}{2}x_t = \frac{3}{2}C_1 - C_2\left(\frac{1}{6}\right)^t$

4.2 Linear systems

Definition 4.3 *If the functions $f_1, \dots, f_n$ in $(*)$ are linear, we obtain a* <u>*linear system*</u>

$$\begin{aligned} x_{1,t+1} &= a_{11}(t)x_{1,t} + \cdots + a_{1n}(t)x_{n,t} + b_1(t) \\ \vdots \quad & \quad \vdots \\ x_{n,t+1} &= a_{n1}(t)x_{1,t} + \cdots + a_{nn}(t)x_{n,t} + b_n(t) \end{aligned}$$

If we define

$$\mathbf{x}_t = \begin{pmatrix} x_{1,t} \\ \vdots \\ x_{n,t} \end{pmatrix}, \ \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix} \ \textit{and}\ \mathbf{b}_t = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

then the system is equivalent to

$$\mathbf{x}_{t+1} = \mathbf{A}(t)\, \mathbf{x}_t + \mathbf{b}_t \qquad (\star)$$

Special cases:

- If <u>$\mathbf{A}(t) = \mathbf{A}$</u> is constant then the linear system $(\star)$ reduces to

  $$\mathbf{x}_{t+1} = \mathbf{A}\, \mathbf{x}_t + \mathbf{b}_t$$

  and we see:

  $$\begin{aligned} \mathbf{x}_1 &= \mathbf{A}\, \mathbf{x}_0 + \mathbf{b}_0 \\ \mathbf{x}_2 &= \mathbf{A}\, \mathbf{x}_1 + \mathbf{b}_1 = \mathbf{A}\, (\mathbf{A}\, \mathbf{x}_0 + \mathbf{b}_0) + \mathbf{b}_1 = \mathbf{A}^2\, \mathbf{x}_0 + \mathbf{A}\mathbf{b}_0 + \mathbf{b}_1 \\ \mathbf{x}_3 &= \mathbf{A}^3\, \mathbf{x}_0 + \mathbf{A}^2\mathbf{b}_0 + \mathbf{A}\mathbf{b}_1 + \mathbf{b}_2 \\ \vdots \quad & \quad \vdots \\ \mathbf{x}_t &= \mathbf{A}^t\, \mathbf{x}_0 + \sum_{k=1}^{t} \mathbf{A}^{t-k}\, \mathbf{b}_{k-1} \end{aligned}$$

- If <u>$\mathbf{A}(t) = \mathbf{A}$ and $\mathbf{b}_t = \mathbf{b}$</u> are constant, we have:

  $$\mathbf{x}_t = \mathbf{A}^t\, \mathbf{x}_0 + (\mathbf{A}^{t-1} + \mathbf{A}^{t-2} + \cdots + \mathbf{A} + \mathbf{I})\, \mathbf{b}$$

  By a direct calculation we see

  $$(\mathbf{A}^{t-1} + \mathbf{A}^{t-2} + \cdots + \mathbf{A} + \mathbf{I})\, (\mathbf{I} - \mathbf{A}) = \mathbf{I} - \mathbf{A}^t$$

  and if $\det(\mathbf{I} - \mathbf{A}) \neq 0$ ($\lambda = 1$ is not an eigenvalue of $\mathbf{A}$), then $(\mathbf{I} - \mathbf{A})^{-1}$ exists and we get

  $$(\mathbf{A}^{t-1} + \mathbf{A}^{t-2} + \cdots + \mathbf{A} + \mathbf{I}) = (\mathbf{I} - \mathbf{A}^t)(\mathbf{I} - \mathbf{A})^{-1}$$

  and

  $$\mathbf{x}_t = \mathbf{A}^t\, \mathbf{x}_0 + (\mathbf{I} - \mathbf{A}^t)(\mathbf{I} - \mathbf{A})^{-1}\, \mathbf{b}$$

- If <u>$\mathbf{A}(t) = \mathbf{A}$ and $\mathbf{b}_t = \mathbf{0}$</u> are constant, we have:

  $$\mathbf{x}_t = \mathbf{A}^t\, \mathbf{x}_0$$

4.3 Stability of linear systems

Definition 4.4 *The linear system* $\mathbf{x}_{t+1} = \mathbf{A}\ \mathbf{x}_t + \mathbf{b}_t$ *is said to be* <u>*globally*</u> <u>*asymptotically stable*</u> *if the general solution of the corresponding homogeneous system* $\mathbf{x}_{t+1} = \mathbf{A}\ \mathbf{x}_t$ *tends to* $\mathbf{0}$ *if* $t \to \infty$.

We see:

$\mathbf{x}_{t+1} = \mathbf{A}\ \mathbf{x}_t + \mathbf{b}_t$ globally asymptotically stable

$\longleftrightarrow$ $\mathbf{x}_t = \mathbf{A}^t\ \mathbf{x}_0 \to \mathbf{0}$ for all $\mathbf{x}_0$

$\longleftrightarrow$ $\mathbf{A}^t \to \mathbf{0}_{n,n}$ (matrix)

$\longleftrightarrow$ all eigenvalues of $\mathbf{A}$ have moduli less than 1

$\longrightarrow$ $(\mathbf{I} - \mathbf{A})^{-1}$ exists!

Theorem 4.2 *If all eigenvalues of* $\mathbf{A}$ *have moduli strictly less than* 1*, the difference equation* $\mathbf{x}_{t+1} = \mathbf{A}\ \mathbf{x}_t + \mathbf{b}$ *is globally asymptotically stable and every solution*

$$\mathbf{x}_t \quad = \quad \mathbf{A}^t\ \mathbf{x}_0 + (\mathbf{I} - \mathbf{A}^t)(\mathbf{I} - \mathbf{A})^{-1}\ \mathbf{b}$$

tends to $(\mathbf{I} - \mathbf{A})^{-1}\ \mathbf{b}$ *if* $t \to \infty$.

Example 4.2 *Investigate the stability of the linear system* $\mathbf{x}_{t+1} = \mathbf{A}\ \mathbf{x}_t$ *with* $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{pmatrix}$.

Solution:

The eigenvalues of $\mathbf{A}$ are 1 and $\frac{1}{6}$, the system is not globally asymptotically stable.

Remark:

We know the general solution already: $\mathbf{x}_t = \begin{pmatrix} A + B\left(\frac{1}{6}\right)^t \\ \frac{3}{2}A - B\left(\frac{1}{6}\right)^t \end{pmatrix}$