Dr. Thomas Zehrt **Topics in Advanced Mathematics** Exercise **1**

Fundamentals

Solve the following problems and hand in your solutions of the ($*$) - marked questions. The solutions will be marked.

Matrices, eigenvalues and eigenvectors

1. Compute the (complex) eigenvalues and the associated (complex) eigenvectors for the following matrices:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

2. ($*$) The rank $r(\mathbf{A})$ of a matrix $\mathbf{A}$ is the maximal number of linearly independent column vectors in $\mathbf{A}$. Determine the rank of the following matrix for all values of the real parameter a. Hint: Use elementary row operations.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 & 0 \\ -1 & a & -1 & a \\ 2 & 0 & 1 & -1 \end{pmatrix}$$

3. ($*$) Verify the **Spectral Theorem for symmetric matrices** for the following matrix by finding a matrix $\mathbf{P}$:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

4. Use the identity $\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$ for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ to prove the following facts:

   (a) $\det(\mathbf{A}^{-1}) = (\det(\mathbf{A}))^{-1}$;

   (b) $\det(\lambda\, \mathbf{A}) = \lambda^n \det(\mathbf{A})$ for $\lambda \in \mathbb{R}$;

   (c) $\mathbf{A}$ and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ have the same eigenvalues.

5. Let $\mathbf{A}$ be a square matrix with $\det \mathbf{A} \neq 0$ and let λ be an eigenvalue of $\mathbf{A}$. Prove:

   (a) $\lambda \neq 0$ and

   (b) $1/\lambda$ is an eigenvalue of the inverse $\mathbf{A}^{-1}$.

6. ($*$) Let $\mathbf{B}$ be an $n \times n$ matrix. Show that $\mathbf{A} = \mathbf{B}^T\mathbf{B}$ is positive semidfinite.

Metrics, norms and inner products

1. (∗) Let $||\mathbf{x}||_p = \sum_{j=1}^n |x_i|^p$ be the p-norm on $\mathbb{R}^n$.

   (a) Determine and draw the set $B := \{\mathbf{x} \in \mathbb{R}^2 \mid ||\mathbf{x}||_p \leq 1\}$ for $p = 1, 2, 0.5$ and ∞.

   (b) Prove that for all $\mathbf{x} \in \mathbb{R}^n$ we have $||\mathbf{x}||_\infty \leq ||\mathbf{x}||_2 \leq \sqrt{n}\,||\mathbf{x}||_\infty$.

2. (∗) Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space.

   (a) Show that $||v|| := \sqrt{\langle v, v\rangle}$ defines a norm on V.

   (b) Show that $d(v, w) := \sqrt{\langle v - w, v - w\rangle}$ defines a metric on V.

3. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of functions continuous in $x_0 \in [a, b]$ with $f_n \rightrightarrows f$. Prove that f (uniform limit) is continuous in x_0.

4. Let $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{n\times n}$,

$$||\mathbf{x}||_\infty = \max_{j=1,\ldots,n} |x_j| \qquad \text{and} \qquad ||\mathbf{A}||_\infty = \max_{\mathbf{x}\neq\mathbf{0}} \frac{||\mathbf{A}\mathbf{x}||_\infty}{||\mathbf{x}||_\infty}$$

   (a) Prove that $||\cdot||_\infty$ is a norm on $\mathbb{R}^n$.

   (b) Prove that $||\cdot||_\infty$ is a norm on $\mathbb{R}^{n\times n}$.

   (c) Prove that $||\mathbf{A}||_\infty \leq \max_{i=1,\ldots,n} \sum_{j=1}^n |a_{ij}|$.

   (d) Prove that $\max_{i=1,\ldots,n} \sum_{j=1}^n |a_{ij}| \leq ||\mathbf{A}||_\infty$.

5. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and

$$||\mathbf{A}||_F := \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{1/2}.$$

   (a) Prove that $||\cdot||_F$ is a norm on $\mathbb{R}^{n\times n}$, the so called Frobenius norm.

   (b) Is $||\mathbf{A}||_F = \max_{\mathbf{x}\neq\mathbf{0}} \frac{||\mathbf{A}\mathbf{x}||_2}{||\mathbf{x}||_2}$?

Functions and Taylor's formula

1. Let $f(x_1, x_2, x_3) = -x_1^2 + 6x_1x_2 - 9x_2^2 - 2x_3^2$. Determine the gradient and the Hesse matrix of f.

2. Prove that if ϕ is twice countinuously differentiable and $\phi(x, y) = c$ defines y as a twice differentiable function of x, then

$$y' = -\frac{\phi_x}{\phi_y} \qquad \text{and} \qquad y'' = -\frac{\phi_{xx} + 2\phi_{xy} \cdot y' + \phi_{yy} \cdot (y')^2}{\phi_y}.$$

3. $(*)$ The following functions are important for economists ($x_1, x_2 \geq 0$):

   - Cobb-Douglas functions: $f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$ with $\alpha_1, \alpha_2 > 0$;
   - CES functions: $f(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$ with $\rho > 0$;
   - Quasilinear functions: $f(x_1, x_2) = x_1 + g(x_2)$ with $g : \mathbb{R} \to \mathbb{R}$ differentiable;
   - Utility function: $f(x_1, x_2) = q_1 \cdot g(x_1) + q_2 \cdot g(x_2)$ with $g : \mathbb{R} \to \mathbb{R}$ differentiable and $q_1 + q_2 = 1$.

   Questions:

   (a) Compute $\frac{f_{x_1}}{f_{x_2}}$ for all functions.

   (b) Compute $\nabla f(\mathbf{a})$ and $\nabla^2 f(\mathbf{a})$, $\det(\nabla^2 f(\mathbf{a}))$ and the sign of $\det(\nabla^2 f(\mathbf{a}))$ for all functions.

   (c) Compute the 2-nd Taylor polynomial $P_2(\mathbf{x}, \mathbf{a})$ of f in $\mathbf{a}$ for all functions.

4. Let $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ be homogeneous of degree d. Prove the following fact: At each point on a given ray through the origin the gradients of f are proportional.

Numerical methods for solving systems of equations

1. ($*$) Let a be a real number, $I = [1,2]$ and

$$F_a(x) \;=\; 1 + a \cdot \left(\frac{1}{x} + \frac{1}{x^2} \right) \;(x > 0).$$

   (a) Show that $F_{0.4}(x)$ is a self mapping on I. Is $F_{0.4}$ a contraction on I?
   (b) Find all real numbers a such that F_a is a self mapping on I.
   (c) Find all real numbers a such that F_a is a self mapping and a contraction on I.

2. ($*$) Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\mathbf{F}(x,y) \;=\; \begin{pmatrix} x^2 + y^2 - 2 \\ y - 1/x \end{pmatrix}.$$

   We are looking for a zero of $\mathbf{F}$.

   (a) Compute $D\mathbf{F}(x,y)$.
   (b) Determine all (x,y) such that $D\mathbf{F}(x,y)$ is not invertible.
   (c) Formulate the Newton method for this problem.
   (d) Compute $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ for $\mathbf{x}^{(0)} = (5/4, 5/4)^T$.
   (e) Compute $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ for $\mathbf{x}^{(0)} = (2,8)^T$.

3. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\mathbf{F}(x,y,z) \;=\; \begin{pmatrix} x + y - xz \\ 2y - yz \\ 0.5 - 0.5x^2 - 0.5y^2 \end{pmatrix}.$$

   We are looking for a zero of $\mathbf{F}$.

   (a) Compute $D\mathbf{F}(x,y,z)$.
   (b) Formulate the Newton method for this problem.
   (c) Compute $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ for $\mathbf{x}^{(0)} = (1,0,0)^T$.

4. Let $\mathbf{F} : D \to \mathbb{R}^n$ be a contraction on D with Lipschitz constant $\lambda < 1$. Prove the so called <u>defect formula</u>:

$$||\mathbf{x} - \mathbf{y}|| \;\leq\; \frac{1}{1-\lambda} \;(\; ||\mathbf{x} - \mathbf{F}(\mathbf{x})|| + ||\mathbf{y} - \mathbf{F}(\mathbf{y})|| \;)$$

   for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and any norm.

Brouwer's fixed point theorem

1. Consider the function f defined on $(0,1)$ by $f(x) = \frac{1}{2}(x+1)$. Prove that f maps $(0,1)$ into itself, but f has no fixed point in $(0,1)$. Why does Brouwer's fixed point theorem not apply?

2. $(*)$ Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ matrix whose elements satisfy

   - $a_{ij} \geq 0$ for all i, j
   - $\sum_{i=1}^{n} a_{ij} = 1$ for all j

   Prove that if $\mathbf{x} \in \Delta^{n-1}$ then $\mathbf{Ax} \in \Delta^{n-1}$. What does Brouwer's fixed point theorem say in this case?

3. Study the paper

   Joel H. Shapiro, Sperner's Lemma and Brouwer's fixed point theorem

Static Optimization

1. The following functions are important for economists ($x_1, x_2 \geq 0$):
   - Cobb-Douglas functions: $f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$ with $\alpha_1, \alpha_2 > 0$;
   - CES functions: $f(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$ with $\rho > 0$;
   - Quasilinear functions: $f(x_1, x_2) = x_1 + g(x_2)$ with $g : \mathbb{R} \to \mathbb{R}$ differentiable;
   - Utility function: $f(x_1, x_2) = q_1 \cdot g(x_1) + q_2 \cdot g(x_2)$ with $g : \mathbb{R} \to \mathbb{R}$ differentiable and $q_1 + q_2 = 1$.

   Let $D = \{(x_1, x_2) \in \mathbb{R}^2 \mid \phi(x_1, x_2) = p_1 x_1 + p_2 x_2 - I\}$ for $p_1, p_2, I > 0$. Compute all stationary points of the corresponding Lagrange functions $l(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda\phi(x_1, x_2)$ for all functions. Use $g(x) = x^k$.

2. (∗) Let $Q(\mathbf{x}) = Q(x_1, x_2) = \mathbf{x}^T \underbrace{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}}_{=\mathbf{A}} \mathbf{x} = ax_1^2 + bx_1x_2 + cx_2^2$

   and $D = \{(x_1, x_2) \in \mathbb{R}^2 \mid \phi(\mathbf{x}) = \mathbf{x}^T\mathbf{x} - r^2 = x_1^2 + x_2^2 - r^2 = 0\}$.

   Prove that Q attains maximum and minimum value over the set D which are related to the largest and smallest eigenvalues of $\mathbf{A}$.

3. Let $\mathbf{A}$ be a symmetric $n \times n$ matrix, $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ a quadratic form and $D = \{\mathbf{x} \in \mathbb{R}^n \mid \phi(\mathbf{x}) = \mathbf{x}^T\mathbf{x} - r^2 = 0\}$. Prove that Q attains maximum and minimum value over the set D which are related to the largest and smallest eigenvalues of $\mathbf{A}$.

4. (∗) Suppose that $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $n \times n$ symmetric positive definite matrix. Show that

   $$Q(\mathbf{y} + t(\mathbf{x} - \mathbf{y})) - tQ(\mathbf{x}) - (1 - t)Q(\mathbf{y}) \leq 0$$

   for all $t \in [0, 1]$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

5. (∗)

   $$\max \quad : f(x, y, z) = \frac{1}{2}x - y$$

   $$\text{subject to} \quad : \begin{cases} g_1(x, y, z) = x - e^{-x} - y + z^2 \leq 0 \\ g_2(x, y, z) = -x \leq 0 \end{cases}$$

6.

   $$\max \quad : f(x, y) = xy + x^2$$

   $$\text{subject to} \quad : \begin{cases} g_1(x, y) = x^2 + y \leq 2 \\ g_2(x, y) = -y \leq 1 \end{cases}$$