Dr. Thomas Zehrt **Topics in Advanced Mathematics** Exercise **2**

Differentiable processes, differential equations and optimal control theory

Solve the following problems and hand in your solutions of the $(*)$ - marked questions. The solutions will be marked.

Differential equations

1. $(*)$ Find the general solutions of the following differential equations:

$$(a) \quad \dot{x} = t^3 - t \qquad (b) \quad \dot{x} + x = 10$$

2. $(*)$ Find the general solutions of the following equations:

$$(a) \quad \ddot{x} - 3x = 0 \qquad (b) \quad \ddot{x} - x = \sin t$$

3. (**Voluntary**) In the theory of option pricing one encounters the equation

$$x^2 f''(x) + axf'(x) + bf(x) = \alpha x + \beta$$

where $f(x)$ denotes the value of a stock option if the price of the stock is x. We assume that $(a-1)^2 > 4b$.

   (a) Find the general solution of the associated homogeneous differential equation. (Hint: $f^h(x) = x^r$)

   (b) Find a particular solution $f^p(x)$ of the differential equation.

Systems of Differential equations

1. $(*)$ Find the general solutions of the following systems:

(a)
$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x \end{aligned}$$

(b)
$$\begin{aligned} \dot{x} &= x - 4y \\ \dot{y} &= 2x - 5y \end{aligned}$$

2. $(*)$ Find the general solution of the following (nonautonomous) system:

$$\begin{aligned} \dot{x} &= 2x - 3y \\ \dot{y} &= -x + t \end{aligned}$$

3. We look at the predator-prey system described by the equations

$$\begin{aligned} \dot{x} &= \alpha\, x - \beta\, x\, y \\ \dot{y} &= \delta\, x\, y - \gamma\, y \end{aligned}$$

with $x = x(t)$ be the number of prey, $y = y(t)$ the number of predator and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ parameters, describing the interaction of the two species.

(a) Prove that $(x^*, y^*) = (\gamma/\delta, \alpha/\beta)$ is a constant solution of the system.

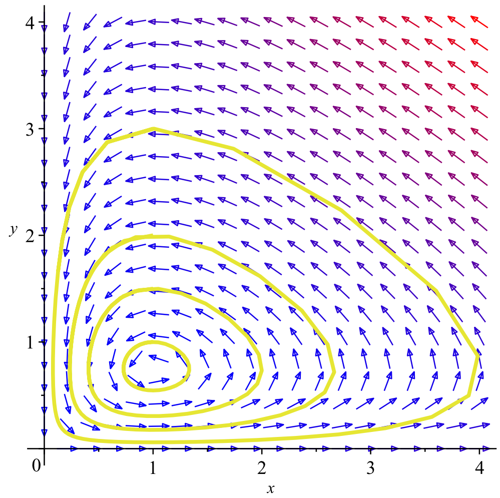
(b) Solve the system if $\beta = \delta = 0$ and $\alpha = \gamma = 1$.

(c) We assume that $x = x(t)$ is an invertible function s.t. $t = t(x)$. Then

$$y'(x) \;=\; \frac{dy}{dx} \;=\; \frac{\dot{y}}{\dot{x}} \;=\; \frac{\delta\, x\, y - \gamma\, y}{\alpha\, x - \beta\, x\, y}.$$

Determine the general (implicit) solution of this (separable) differential equation for $y = y(x)$.

(d) (**Voluntary**) Use Maple to plot the phase diagrams with some solution curves for $\alpha = 0.3$, $\beta = 0.4$, $\gamma = 0.3$ and $\delta = 0.3$.



(e) Use $f(x, y) = x \cdot y \approx x_0 \cdot y_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ to linearize the system near (x_0, y_0). Prove that the general linearized system can be written as

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha - \beta y_0 & -\beta x_0 \\ \delta x_0 & \delta x_0 - \gamma \end{pmatrix}}_{=A} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + x_0 y_0 \begin{pmatrix} \beta \\ -\delta \end{pmatrix}.$$

(f) Determine the linearized system near $(x_0, y_0) = (x^*, y^*) = (\gamma/\delta, \alpha/\beta)$ and solve this system.

We can try to control the (uncontrolled) predator-prey system in various ways. If the prey is a pest then we could use an ideal pesticide (agent that kills only the pests in a density-dependent manner). At time t, let the rate of application of the pesticide be $u = u(t)$. The dynamics of the new (controlled) predator-prey system subject to this control variable is given by

$$\begin{aligned} \dot{x} &= \alpha\, x - \beta\, x\, y - \epsilon\, u\, x \qquad (\epsilon > 0) \\ \dot{y} &= \delta\, x\, y - \gamma\, y. \end{aligned}$$

We know that the point $P = (\gamma/\delta, \alpha/\beta)$ has the nice property that the pest population remains at a constant level. A control problem could be: How should we use the pesticide (determine $u(t)$) to transfer the system to P (as fast as possible).

Calculus of variations

1. (∗) Find the Euler equation for the following problem and the solution of the associated initial value problem:

$$\max \int_0^1 (4ut - \dot{u}^2)\, dt, \qquad u(0) = 2,\ \ u(1) = 2/3.$$

2. (∗) Find the Euler equation for the following problem:

$$\max \int_{t_0}^{t_1} (u^2 + \dot{u}^2 + 2ue^t)\, dt.$$

3. Consider an economy evolving over time where $K = K(t)$ denotes the capital stock, $C = C(t)$ the consumption and $Y = Y(t)$ the net national product at the time t. Suppose that $f(K) = Y$ where $f'(K) > 0$ (strictly increasing) and $f''(K) < 0$.

   - For each time assume that
     $$f(K(t)) \;=\; C(t) + \dot{K}(t) \tag{1}$$
     which means that output $Y(t) = f(K(t))$ is divided between consumption $C(t)$ and investment $\dot{K}(t)$.
   - Let $K(0) = K_0$ be the given capital stock existing today at $t = 0$ and suppose that there is a fixed planning period $[0, T]$.
   - For each choice of investment function $\dot{K}(t)$ on the interval $[0, T]$ the capital is determined by
     $$K(t) \;=\; K_0 + \int_0^t \dot{K}(\tau)\, d\tau \tag{2}$$
     and (1) determines $C(t)$.
   - Assume that the society has a utility function U, where $U = U(C)$ is the utility flow the country enjoys when the total consumption is C. Suppose $U'(C) > 0$ and $U''(C) < 0$.
   - Introduce a discount rate r to reflect the idea that the present may matter more than the future. For each $t \geq 0$ multiply $U(C(t))$ by the discount factor e^{-rt}.

   The goal of investment policy is to choose $K(t)$ for $t \in [0, T]$ in order to make the total discounted utility over the period as large as possible:

   Find the function $K = K(t)$, with $K(0) = K_0$ that maximizes

   $$\int_0^T U(C(t)) \cdot e^{-rt} dt \;=\; \int_0^T U(\, f(K(t)) - \dot{K}(t)\,) \cdot e^{-rt} dt \tag{3}$$

   (a) Find the Euler equation and solve the variation problem

   $$\max \int_0^T \ln(2K - \dot{K}) \cdot e^{-t/4}\, dt, \qquad K(0) = K_0,\ \ K(T) = K_T$$

   (b) Find the Euler equation and solve the variation problem

   $$\max \int_0^{10} \left(\frac{1}{100}\, t\, K - \dot{K}^2 \right) \cdot e^{-t/4}\, dt, \qquad K(0) = 0,\ \ K(10) = 20$$

   (c) Find the (general) Euler equation for the Ramsey problem (3).