Dr. Thomas Zehrt | **Topics in Advanced Mathematics** | Exercise **3**

Discrete processes, difference equations and discrete time optimization

Solve the following problems and hand in your solutions of the $(*)$ - marked questions. The solutions will be marked.

Difference equations

1. $(*)$ Find the general solution and the particular solution such that $y_0 = 3$ and $y_1 = 2$ of the difference equation

$$y_{t+2} - 6y_{t+1} + 8y_t \quad = \quad 0.$$

2. Investigate the global asymptotic stability of the following difference equations:

$$(a) \qquad y_{t+2} - y_{t+1} - y_t \quad = \quad 0$$

$$(b) \quad y_{t+2} - \frac{1}{8}y_{t+1} + \frac{1}{6}y_t \quad = \quad t^2 e^t$$

3. Find the general solution of the difference equation

$$y_{t+3} - 3y_{t+2} + 3y_{t+1} - y_t \quad = \quad 24(t+2).$$

4. $(*)$ Solve the (nonlinear) difference equation $y_{t+1} - y_t + t\, y_{t+1}\, y_t \; = \; 0.$
(Hint: Divide by $y_{t+1}\, y_t$)

5. Solve the (nonlinear) difference equation $y_{t+2}\, y_t^2 = y_{t+1}^3$. (Hint: ln)

6. A and B play a game. In each step of the game A can win a penny from B with probability p while B can win a penny from A with probability $q = 1-p$. We assume that no tie can occur. Suppose that the game is started with A and B having a and b pennies respectively, the total being $a + b = N$, and that the game ends when one or the other has M pennies. p_k denotes the probability that A wins the game when he has k pennies.

   (a) Formulate a difference equation for p_k.

   (b) Formulate boundary conditions for p_k.

   (c) Obtain the probability p_a that A wins the game, if

   - $p \neq q$ and
   - $p = q = \frac{1}{2}$.

Discrete time optimization

1. (∗) Use the fundamental equations of dynamic programming to solve the problem

$$\max \sum_{t=0}^{2} \left(1 - (x_t^2 + 2u_t^2) \right)$$
$$\text{subject to}$$
$$\begin{aligned} x_{t+1} &= x_t - u_t, \\ x_0 &= 5, \\ u_t &\in \mathbb{R}. \end{aligned}$$

   Compute $J_s(x)$ and $u_s^*(x)$ for $s = 2, 1, 0$.

2. Consider the problem

$$\max \sum_{t=0}^{T} \left(\frac{1}{1-r} \right)^t \cdot \sqrt{u_t x_t}$$
$$\text{subject to}$$
$$\begin{aligned} x_{t+1} &= \rho(1 - u_t)x_t, \\ x_0 &> 0, \\ u_t &\in [0, 1] \subset \mathbb{R} \end{aligned}$$

   where r is the rate of discount. Compute $J_s(x)$ and $u_s^*(x)$ for $s = T, T-1, T-2$.