# The End of Civilization:
## Introduction to dynamical systems
*on the example of Easter Island*

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# Lecture Plan

General Introduction to dynamical systems

Easter Island civilization&Malthus

Richardo-Malthus model

Application to Easter Island's problem

Concluding remarks

# What is the dynamical system?

- Includes time as an independent variable
- Tracks changes of some function over time
- Includes one or more **motion laws**
- Can be discrete, continuous, hybrid, delayed...
- Describes numerous natural and social phenomena in time

# What is the dynamical system?

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- Describes numerous natural and social phenomena in time

- **Solution** of dynamical system is explicit function of time.

# Discrete-time systems

Discrete-time dynamical system of dimension 1 may be represented as the iteration of a real-valued function:

$$x_n = f(x_{n-1});$$
$$n \in \mathbb{Z}.$$

Then the **trajectory** may be represented as a sequence:

$$x_1 = f(x_0)$$
$$x_2 = f(x_1) = f(f(x_0)) = f^2(x_0)$$
$$x_3 = f(x_2) = f(f(x_1)) = f(f(f(x_0))) = f^3(x_0)$$
$$\ldots$$
$$x_n = f^n(x_0)$$

# Fixed and periodic points

### Definition

A point $\bar{x}$ is a **periodic point of period** $k$ provided
$f^k(\bar{x}) = \bar{x}$
and
$f^k(\bar{x}) \neq \bar{x}$ for $0 < j < k$.
A periodic point with period $k = 1$ is a **fixed point**

- **Periodic orbit**: A solution which visits $\bar{x}$ every $k$ periods
- **Steady state**: An orbit consisting solely of $\bar{x}$: constant orbit.

Example of fixed point analysis

# Continuous-time systems

The continuous-time dynamical system is given by the **law of motion**:

$$\frac{d}{dt}x(t) = \dot{x} = f(x, t); \; x \in D \subset \mathbb{R}^n, t \in \mathbb{R}$$

- To find the **solution** is to find explicit function $x(t) = \Phi(t, x)$ called the **evolution function**
- The solution does not exist always and can be non-unique

# Classification

- Dynamical system is **autonomous** if $f(x,t) = f(x)$, the equation does not explicitly depend on time
- Dynamical system is **linear**, if $f(t,x) = A(t)x + B(t)$, it is linear in the state variable
- Dynamical system is **finite-dimensional**, if $\dim x < \infty$
- Dynamic system is an ODE system, if its solution is $x(t) = \Phi(t)$ - function of time only

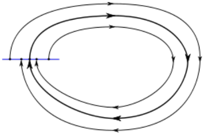
Linear autonomous finite-dimensional ODE always has a unique solution.

# Fixed points and limit cycles

- Point $x(t)$ is periodic if exists such $t_0$ that

$$x(t + t_0) = x(t)$$

- The trajectory is **closed** or **cycle** if it returns to the starting point:



  - A **fixed point** is such periodic point that $t_0 = 0$.
- To find the fixed point is to find a **steady state** of the dynamical system

# Linear autonomous systems: Stability

Let the dynamical system be given by

$$\dot{x} = Ax;$$
$$x \in \mathbb{R}^n.$$

Then the only fixed point (and solution) of the system is $\bar{x} = 0$.

- If all eigenvalues of $A$ have negative real parts, then every solution is stable($\bar{x}$ is a sink)
- If any eigenvalue of $A$ have positive real part, then every solution is unstable ($\bar{x}$ is a source)
- If some of the eigenvalues of $A$ have zero real parts and all other have negative real parts, then let
  $\lambda = i\sigma_1, i\sigma_2, .., i\sigma_m$
  be eigenvalues with zero real parts. If the multiplicity of all such eigenvalues is one, then every solution is stable.

Example of steady state analysis

# Classification of stability regimes for 2-dim systems

| Type | $Re(\lambda)$ | $Im(\lambda)$ |
|---|---|---|
| Source node | $> 0$ | $= 0$ |
| Sink node | $< 0$ | $= 0$ |
| Saddle | $\lambda_1 > 0, \lambda_2 < 0$ | $= 0$ |
| Center | $= 0$ | $\neq 0$ |
| Spiral source | $> 0$ | $\neq 0$ |
| Spiral sink | $< 0$ | $\neq 0$ |

- **Multiplicity** of eigenvalues further specifies the dynamics
- For higher-dimensional systems classification is more complex
- For non-linear systems stability is studied by the **linearization**

# Stability for non-linear (autonomous) ODE systems

Given an ODE system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$

The following algorithm is usually applied:

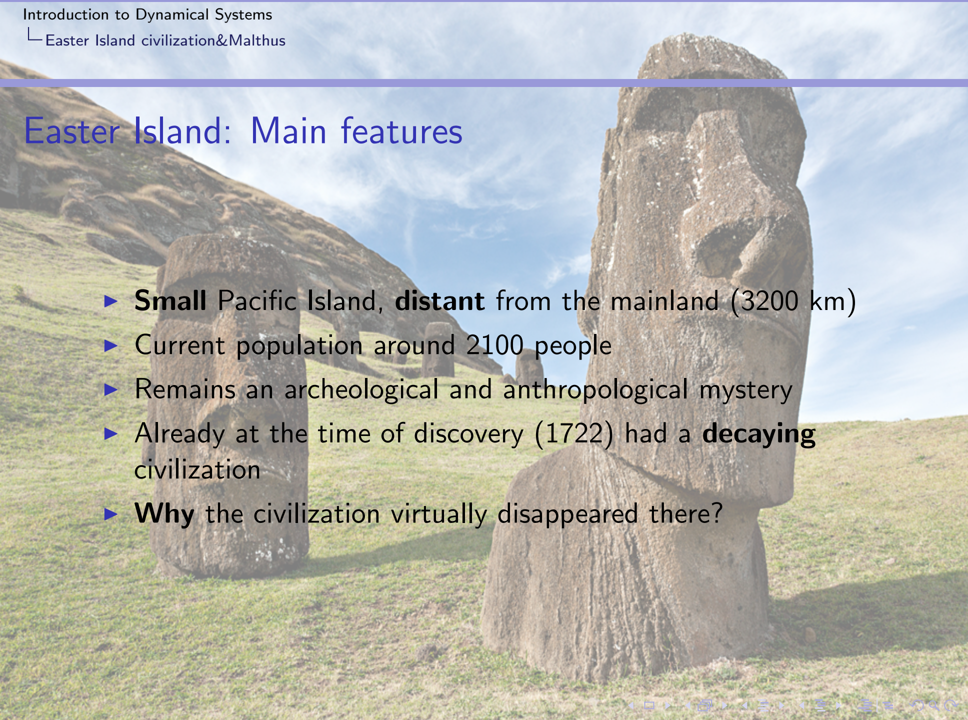
- Define the **Jacobian** of a system as:

$$\mathbf{J} \overset{def}{=} \begin{pmatrix} \frac{\partial \mathbf{F}_1(\mathbf{x})}{\partial x_1} & \frac{\partial \mathbf{F}_1(\mathbf{x})}{\partial x_2} & \dots & \frac{\partial \mathbf{F}_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial \mathbf{F}_2(\mathbf{x})}{\partial x_1} & \frac{\partial \mathbf{F}_2(\mathbf{x})}{\partial x_2} & \dots & \frac{\partial \mathbf{F}_2(\mathbf{x})}{\partial x_n} \\ \dots & \dots & \dots & \\ \frac{\partial \mathbf{F}_n(\mathbf{x})}{\partial x_1} & \frac{\partial \mathbf{F}_n(\mathbf{x})}{\partial x_2} & \dots & \frac{\partial \mathbf{F}_n(\mathbf{x})}{\partial x_n} \end{pmatrix}$$

- Compute eigenvalues of an associated linearized system

$$\dot{\mathbf{u}} \overset{def}{=} \mathbf{J}(\mathbf{x}^*)\mathbf{u} + R(\mathbf{x})$$

- Stability for original system around steady states is equivalent to this one.



# Easter Island: Main features

- **Small** Pacific Island, **distant** from the mainland (3200 km)
- Current population around 2100 people
- Remains an archeological and anthropological mystery
- Already at the time of discovery (1722) had a **decaying** civilization
- **Why** the civilization virtually disappeared there?

# Malthus and overpopulation

- Resources are limited
- Population is growing
- Eventually the overpopulation will lead to stagnation, starvation and collapse
- Remedy: less population, less consumption: **Limits to Growth**.

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**BUT:** We do not want to consume less and want to breed.

- How we can **sustainably use resources and still evolve?**

# Renewable resources and sustainability concept

- Resource grows at some rate
- Population uses the resource for activities at growing rates also
- Usage is **sustainable**, if:
  1. The resulting dynamical system has a steady state
  2. This steady state admits non-zero resource and population
  3. This steady state is (at least) saddle-type stable
  4. Steady state may be supported infinitely (sufficiently) long

Now think of modern situation with resource usage:
**Is it sustainable?**

# Main ingredients

- Limited resource stock $S(t)$ with regeneration is harvested
- Growing population $L(t)$ is employed for harvesting and other good $M$ production
- Dynamical 2-d system
- Steady-states analysis and transitional dynamics

# Renewable resource dynamics

- Resource grows through time by $G(S)$
- It is consumed at rate $H(t)$ (**harvest**)
- Growth law is logistical with $K$ the **carrying capacity**:

$$G(S) = rS(t)(1 - S(t)/K)$$

giving the dynamics of the **resource stock**:

$$\frac{dS}{dt} \stackrel{def}{=} \dot{S} = rS(t)(1 - S(t)/K) - H(t) \tag{1}$$

# Harvesting function

- Harvest is consumed
- To produce the harvest labour input plus resource input are required (production function):

$$H^p = \alpha S L_H$$

- Price of resource good equals its costs of production:

$$p = w a_{LH} = \frac{w}{\alpha S}$$

# Temporary Ricardian equilibrium

- Individual utility:

$$u = h^{\beta} m^{1-\beta}$$

- Total demand:

$$H^D = w\beta L/p;\ M^D = w(1-\beta)L$$

- Full employment:

$$H^P a_{LH}(S) + M = L$$

Lead to **Ricardian equilibrium**:

$$H = \alpha\beta LS$$

# Malthusian population dynamics

- Consumption of resource increases fertility
- Otherwise growth is fixed:

$$\frac{dL}{dt} \stackrel{def}{=} \dot{L} = L(t)[b - d + F(t)] \qquad (2)$$

with fertility function

$$F(t) = \phi H(t)/L(t) \qquad (3)$$

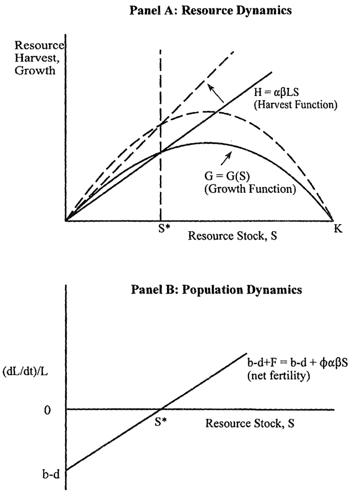
# Dynamical system

The intertemporal dynamics consists of:

- Population dynamics, (2)
- Resource stock dynamics,(**??**)
- Yielding the 2-dim ODE system:

$$\begin{aligned} \dot{L} &= L(t)[b - d + \phi\alpha\beta S(t)] \\ \dot{S} &= rS(t)(1 - S(t)/K) - \alpha\beta L(t)S(t) \end{aligned} \tag{4}$$

which is non-linear.




A RICARDO-MALTHUS STEADY STATE

## Steady states

These are given by solutions to

$$\begin{cases} \dot{L} = 0 \\ \dot{S} = 0 \end{cases}$$

- There are 3 steady states in total:
- One **interior**:

$$\bar{L}_2 = \frac{r}{\alpha\beta}\left(1 - \frac{d-b}{\phi\alpha\beta K}\right);$$
$$\bar{S}_2 = \frac{d-b}{\phi\alpha\beta}$$

- Two **corner**:

$$\bar{S}_1 = 0, \bar{L}_1 = 0$$
$$\bar{S}_3 = K, \bar{L}_3 = 0$$

# Dynamics

- System cannot be explicitly solved;
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- **Local** behavior: stability of steady states;
- **Global** behavior: which steady state is reached from which initial states.

# Local stability

- The interior steady state exists only if

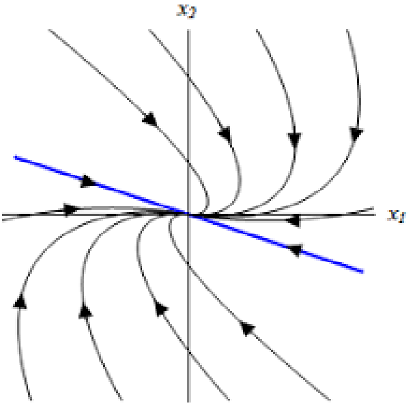
$$(d - b)/(\phi\alpha\beta) < K \tag{5}$$

- $\bar{S}_1, \bar{L}_1$ is a saddle with $S = 0$ **stable manifold**
- $\bar{S}_3, \bar{L}_3$ is a saddle-point with $L = 0$ stable manifold
- Interior steady state is stable and either:
  - Spiral node
  - Improper node

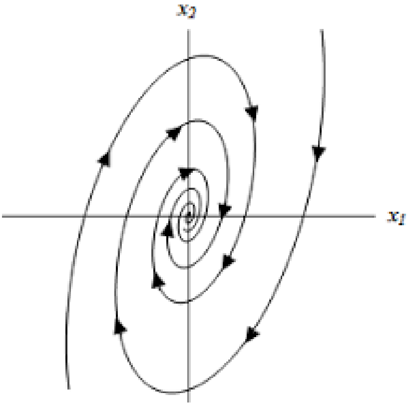
Exemplary calculations

# Illustration

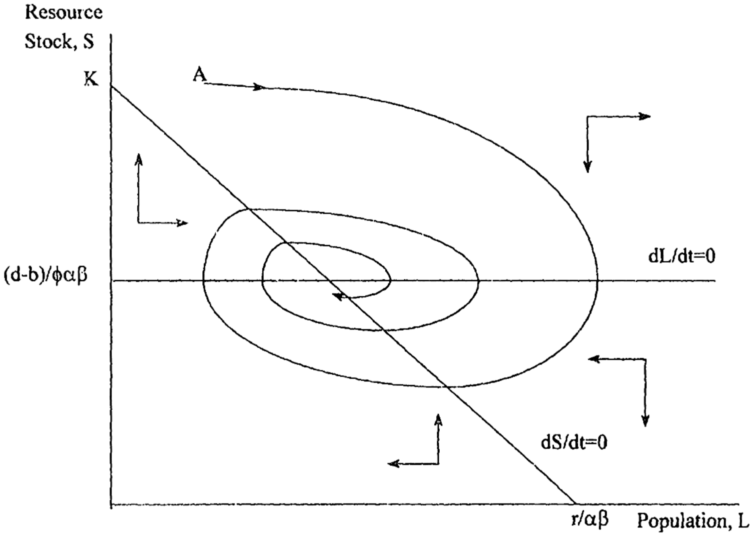
Improper Node - Asymptotically Stable




Spiral - Asymptotically Stable



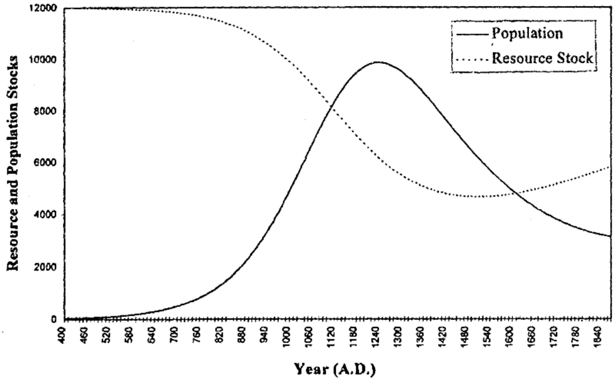
# Global dynamics



# Parameters choice and interpretation

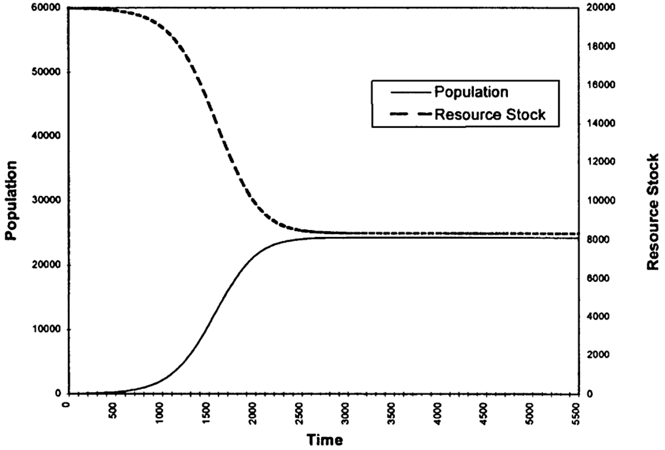
- Carrying capacity is forest stock;
- Time in 10-years periods;
- $b - d = -0.1$: Population will die out without the resource;
- Resource-based good is less preferred than manufactured one;
- Calibration defines the dynamics of a system

# Easter Island case: dying out




EASTER ISLAND BASE CASE

# Sustainable case: aka BGP dynamics




POPULATION AND RESOURCE DYNAMICS WITH FAST REGENERATION

# Easter Island: Possible Answer

- Easter Island not very different from other Polinesian islands;
- The **regeneration rate** of the resource is crucial;
- This translates to different palm species (slow growing) at Easter Island!
- Known as population-resource overshooting

# Place of Dynamics in Resource Economics

- Dynamics is **essential**
- Not the stock, but extraction rates are important (i. e. dynamic quantities)
- It thus has to be **controlled**
- Optimal management of resource usage is necessary
- Improper usage may lead to conflicts, population decrease, etc..
- Time horizon and periods are important
- Calibration from empirical research also very important

## Take home message

Resource management is dynamic.
Improper management may have dire consequences.
Stability of equilibrium is important.

# Next week:

- Dynamic optimization concept: Ramsey (1927)
- Microperspective: optimal resource management
- Introduction to optimal control theory
- Time to extraction and overexploitation
- Paper: Hotelling H. (1931) The Economics of Exhaustible Resources. *The Journal of Political Economy, 39(2)*, pp. 137-175

# Example of a fixed point analysis

Let

$$f(x) = x^3 - x \tag{6}$$

Dynamical system is given then by

$$x_n = x_{n-1}^3 - x_n \tag{7}$$

Fixed points satisfy

$$\bar{x} = \bar{x}^3 - \bar{x} \tag{8}$$

Providing three fixed points:

$$\bar{x} = \{0, \pm\sqrt{2}\}. \tag{9}$$

# Stability for 2-dim. linear systems

- The 2-dim. systems' stability is analyzed through the 2-dim. matrix (easy);
- Given system

$$\dot{x} = ax + by + C_1 \tag{10}$$
$$\dot{y} = cx + dy + C_2. \tag{11}$$

- System matrix is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{12}$$

- Eigenvalues are obtained as roots of the **characteristic polynomial**

$$p(\lambda) = \det(A - \lambda I) = 0 \tag{13}$$

## 2-dim Stability cont'd

- This system has exactly 2 **eigenvalues**;
- The **real part**, $Re(\lambda)$ defines the stability;
- The **imaginary part**, $Im(\lambda)$ defines the type of fluctuations.
- Now recall that

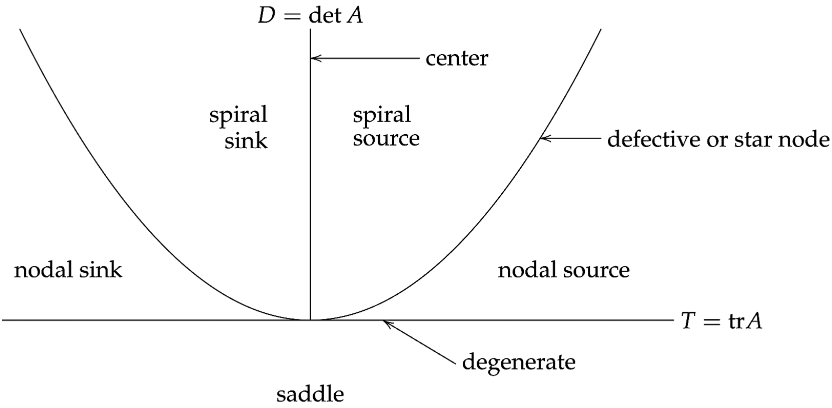
$$tr(A) = a + d, \det(A) = ad - bc, \quad (14)$$

$$p(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc). \quad (15)$$

- This gives characterization of stability through trace and determinant only:

$$\lambda_{1,2} = \frac{-tr(A) \pm \sqrt{tr^2(A) - 4\det(A)}}{2} \quad (16)$$

# Trace-determinant diagram for stability of 2-dim. systems



# Stability of steady-states in the model

- Derivation of (5):

$$\dot{L} = 0,\ L \neq 0 \rightarrow b - d + \phi\alpha\beta S = 0 \rightarrow S = \frac{d-b}{\phi\alpha\beta} \rightarrow (5)$$

- Stability defined through Jacobian matrix:

$$J_{11} = (b-d) + \phi\alpha\beta S;\ J_{12} = \phi\alpha\beta L;$$
$$J_{21} = -\alpha\beta S;\ J_{22} = r - 2rS/K - \alpha\beta L$$

with $\mathbf{J}(0,0) = \begin{pmatrix} b-d & 0 \\ 0 & r, \end{pmatrix}$ giving
$\lambda_{1,2}(\mathbf{J}(0,0)) = \{b-d<0;\ r>0\}$ a saddle-point

Back