Basic models: Intertemporal optimization and resource extraction

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Plan of the lecture

- Idea of intergenerational optimization: Ramsey leads to Hotelling
- Introduction to optimal control theory
- Optimal extraction without regeneration: Hotelling-type problems

Ramsey

- Idea of optimization of consumption over infinite time
- Introduction of discount rates
- Dynamic choice between consumption c and accumulation k
- Goal of maximizing utility over time:

$$\max_c \sum_{t=0}^{\infty} \mathrm{e}^{-\rho t} U(c)$$

- While consumption is taken from output:

$$c_t = Y_t - k_t$$

- Output grows through investments which are necessary for growth:

$$k_{t+1} = Y_t(1 - c_t) - \delta k_t$$

Modern formulation: Ramsey-type problems

$$J := \int_0^\infty \mathbf{e}^{-rt} U(c) dt \rightarrow \max_c$$

$s.t.$

$$\dot{k} = f(k) - c - (n + \delta)k$$

$$k(0) = k_0$$

$$0 \leq c \leq f(k)$$

This is an **optimal control problem** with one state variable and one control variable.

From Ramsey to Hotelling: why we need optimal control

- We need to optimize investments and consumption over **many time periods**
- **Exhaustible resources** enter production function, and not only capital
- Dynamic problem of optimal rate of resource exploitation
- **Sustainability** concept has been born
- Hotelling (1931) was the first to apply Ramsey-type analysis to exhaustible resources management.

Main features

- Multi-stage decision-making
- Optimization of a dynamic process in time
- Optimization is carried over **functions**, not **variables**
- The planning horizon of an optimizing agent is taken into account (finite or infinite)
- The problem includes **objective** and the **dynamical system**
- Some initial and/or terminal conditions are given.

Continuous-time problems

- Assume there is **continuous** number of stages (real time)
- State is described by continuous time function, $x(t)$
- Initial and terminal states are fixed, $x(0) = x_0, x(T) = x_T$
- Find a function $x(t)$, minimizing the cost of going from x_0 to x_T
- What gives the costs: Concept of **objective functional**:

$$\min_u \int_0^T \{x(t) + u^2(t)\}\, dt$$

- Control is chosen to change the state **trajectory** optimally.

Ingredients of dynamic optimization problem

Every dynamic optimization problem should include:

1. **Boundary conditions**: fixed starting and/or terminal points
2. Description of **admissible paths** from initial point to the terminal one: which trajectories are allowed
3. **Costs**, associated with different paths
4. An **objective**: what to maximize or minimize
5. **Dynamic constraints**: the motion law for state of the system (controlled or uncontrolled)

Functionals

Definition

A **functional** J is a mapping from the set of paths $x(t)$ into real numbers (value of a functional).
$J := J(x(t))$.

- Functional is NOT a function of t;
- $x(t)$ is the *unknown* function, which have to be found;
- This is defined in some *functional space* $\mathcal{H}$;
- Hence formally $J : \mathcal{H} \to \mathbb{R}$.

Types of boundary conditions

1. **Fixed-time problem**: $x(0) = x_0$, time length is fixed to $t \in [0, T]$, terminal state $x(T)$ is not fixed
   - Optimal price setting over fixed planning horizon
2. **Fixed endpoint problem**: $x(0) = x_0, x(T) = x_T$ and T is free
   - Cost minimization given required output x_T
3. **Time-optimal problem**: $x(0) = x_0, x(T) = x_T$, $T \to \min$
   - Producing a product as soon as possible regardless of the costs;
4. **Terminal surface problem**: $x(0) = x_0$, and at terminal time $f(T) = x(T)$.

Transversality

- In variable endpoint problems as above given boundary conditions are not sufficient to find the optimal path
- Additional condition on trajectories is called **transversality** condition:
- It defines, how the trajectory crosses the boundary line
- The vast majority of economic problems use this type of conditions
- Example: (discounted) shadow costs of investments $\psi(t)$ at the terminal time should be zero

$$\lambda(T) = 0,\ \mathrm{e}^{-rT}\lambda(T) = 0,\ \lim_{t\to\infty}\lambda(t) = 0,\ \lim_{t\to\infty}\mathrm{e}^{-rt}\lambda(t) = 0$$

Problem

Maximize (minimize) some objective functional

$$J = \max_{u(\bullet)} \int_0^T F(x(t), u, t)dt$$

with conditions on:

- Initial, terminal states and time;
  $x(0) = x_0; x(T) = x_T, t \in [0, T]$
- Dynamic constraints (define the dynamics of states);
  $\dot{x}(t) = f(x, u, t)$
- Static constraints on states (nonnegativity, etc.)
  $x(t) \geq 0, u(t) \geq 0.$

Hamiltonian

- To solve an optimal control problem we need **Hamiltonian function**
- This is an analog of Lagrangian for static problems
- Composition of Hamiltonian:
  1. Objective
  2. Each dynamic constraint times **co-state**
  3. Each static constraint times **dual**
- **IMPORTANT**: Duals differ from co-states, they do not have dynamics!
- First order conditions on Hamiltonian provide optimality criteria

Construction

Let the optimal control problem be:

$$J := \int_0^T F(x,u,t)dt \to \max_u;$$

$$s.\ t.$$

$$\dot{x} = f(x,u,t) \tag{1}$$

Then the associated Hamiltonian is given by:

$$\mathcal{H}(\lambda, x, u, t) = F(x,u,t) + \lambda(t) \cdot f(x,u,t). \tag{2}$$

Comments

- In the Hamiltonian $\lambda(t)$ is called costate variable
- It usually represents shadow costs of investments
- Investments are the control $u(t)$
- It does not have to be continuous: investment may jump
- Hamiltonian includes that many costate variables as many dynamic constraints the system has
- Costate variable changes in time via the **co-state equation** given by derivative w. r. t. the state
- The optimal dynamics is defined by ODE system: for **state**, $x(t)$ and **costate**, $\lambda(t)$.

Example

Consider the problem:

$$\max_{u(\bullet)} \int_0^T e^{-rt}[x(t) - \frac{\alpha}{2}u(t)^2]dt$$

$$\begin{aligned} &s.t. \\ &\quad \dot{x}(t) = \beta(t) - u(t)\sqrt{x(t)}, \\ &\quad u(t) \geq 0, x(0) = x_0. \end{aligned} \tag{3}$$

where $\beta(t)$ is arbitrary positive-valued function and α, r, T are constants.

The Hamiltonian of the problem (3) should be:

$$\mathcal{H}(\lambda, x, u, t) = e^{-rt}[x - \frac{\alpha}{2}u^2] + \lambda[\beta(t) - u\sqrt{x}] \tag{4}$$

- The **admissible set** of controls include all nonnegative values $(u(t) \geq 0)$
- Transformation
  $e^{-rt}\lambda(t) = \lambda^{CV}(t)$ yields
  current value Hamiltonian:

$$\mathcal{H}^{CV}(\lambda, x, u, t) = [x - \frac{\alpha}{2}u^2] + \lambda^{CV}[\beta(t) - u\sqrt{x}]$$

It is used throughout all the economic problems.

Optimality

To obtain first-order conditions of optimality the *Pontryagin's Maximum Principle* is used.
Let

$$J := \int_0^T F(x, u, t)dt \to \max_u;$$
$$s.t.$$
$$\dot{x} = f(x, u, t);$$
$$u(t) \geq 0, x(0) = x_0. \quad (5)$$

be the optimal control problem and

$$\mathcal{H}(\lambda, x, u, t) = F(x, u, t) + \lambda(t) \cdot f(x, u, t). \quad (6)$$

the associated Hamiltonian.

Optimality conditions

The **optimal** control $u(t)$ is such that it maximizes the Hamiltonian, (6) in such a way that

$$u^* : \frac{\partial \mathcal{H}(\lambda, x, u, t)}{\partial u} = 0;$$
$$\mathcal{H}(\lambda, x, u^*, t) = \mathcal{H}^*(\lambda, x, t) \qquad (7)$$

must hold for *almost all* t.
This is **maximum condition**.
Along optimal trajectory

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}^*(\lambda, x, t)}{\partial x}. \qquad (8)$$

which is the **co-state equation**.

$$\lambda(T) = 0 \qquad (9)$$

is transversality condition.

Theorem (Pontryagin's Maximum Principle)

Consider the optimal control problem (5) *with Hamiltonian* (6) *and maximized Hamiltonian as above.*
Assume the state space X is a convex set and there are no scrap value.
Let $u(\bullet)$ be a feasible control path with associated state trajectory $x(t)$.
If there exists and absolutely continuous function $\lambda : [0, T] \to \mathbb{R}^n$ such that
the maximum condition (7)*, the adjoint equation* (8) *and transversality condition* (9) *are satisfied,*
and such that the function $x \to \mathcal{H}^*(\lambda, x, t)$ is concave and continuously differentiable,
then *$u(\bullet)$ is an optimal control.*

Sufficiency

- The theorem provides only necessary, but not sufficient criteria of optimality;
- The sufficient condition is given by the **concavity**(convexity) of a maximized (minimized) Hamiltonian $\mathcal{H}^*$ w. r. t. $x(t)$;
- In our example the Hamiltonian is linear in state and quadratic in control, and hence concave in state;
- Sufficient condition is satisfied;
- This is always true for **linear-quadratic** problems.

Main points on optimal control

Solution of optimal control problem:

1. Write down the Hamiltonian of the problem
2. Derive first-order condition on the control
3. Derive co-state equation
4. Substitute optimal control candidate into state and co-state equations
5. Solve the canonical system of equations (state and co-state)
6. Define optimal control candidate as a function of time
7. Determine the concavity of a maximized Hamiltonian

Social optimum

- Assume the utility depends on resource only
- Social planner maximizes utility subject to finiteness of the resource:

$$\max_{R(\bullet)} \int_0^T \mathrm{e}^{-rt} \left\{ U(R(t)) - \bar{b}R(t) \right\} dt$$
$$s.t.$$
$$\int_0^T R(t)dt = S_0. \tag{10}$$

- After some manipulation this constitutes the standard optimal control problem.

Canonical form

- We set $S(t)$ the new artificial state variable:

$$\dot{S} = -R(t)$$

- Adjoin this new dynamic constraint to the optimization problem (10)
- Transformation from extraction rates R to stocks S yields canonical form of the problem

Hotelling in optimal control form

- Transformation above yields:

$$\max_{R(\bullet)} \int_0^T \mathrm{e}^{-rt} \left\{ U(R(t)) - \bar{b}R(t) \right\} dt$$
$$s.t.$$
$$\dot{S} = -R(t). \tag{11}$$

with S being the state and R the control

- Giving the **Hotelling's rule**:

$$U'(R(t)) - \bar{b} = \lambda(t)\mathrm{e}^{rt} \tag{12}$$

OR

$$\frac{\dot{p}(t)}{p(t)} = \frac{\dot{\lambda}(t)}{\lambda(t)} = r \tag{13}$$

Competitive problem

- There exist n identical resource-extracting firms;
- The optimization problem is then:

$$\max_{R(\bullet)} \int_0^T \mathrm{e}^{-rt} \left\{ \int_0^{nR(t)} p(x)dx - nb(R(t), S(t)) \right\} dt$$

$$s.t.$$

$$n\dot{S} = -nR(t),\ S(0) = \bar{S},\ S(T) = S_T(= 0). \tag{14}$$

Monopolistic problem

- There exists a monopoly on the resource;
- Objective is profit maximization:

$$\max_{R(\bullet)} \int_0^T \mathrm{e}^{-rt} \left\{ p(R)R(t) - b^{MONO}(R(t), S(t)) \right\} dt$$
$$s.t.$$
$$\dot{S} = -R(t). \tag{15}$$

Time to resource depletion

- Parameter T above is free;
- It is defined as an additional choice variable in optimal control problem;
- The condition for that is zero optimized Hamiltonian value:

$$\mathcal{H}(R^*(T^*), S^*(T^*), \lambda^*(T^*), T^*) = 0. \tag{16}$$

Conclusion

- Resource optimization is essentially intertemporal, i. e. dynamic;
- Finite resource requires social planning in most cases;
- With some parameters market can follow optimal path (less frequent than not);
- Intertemporal discount rate influences heavily the outcome;
- Optimal control is a versatile tool for solving intertemporal resource extraction problems.

References

- Ramsey F. (1928) A Mathematical Theory of Saving. *The Economic Journal, Vol. 38 No. 152 pp. 543-559*;
- Hotelling H. (1931) The Economics of Exhaustible Resources. *Journal of Political Economy, Vol. 39, No. 2 pp. 137-175*;
- Seierstad A., Sydsaeter K. *Optimal Control Theory with Economic Applications.* Elsevier, 1999.

Next time

- Extension of optimal resource management to renewable resources
- Water management framework
- Sustainable usage of renewable resources
- Paper: Gisser M. and Sanchez D. (1980) Competition versus optimal control in groundwater pumping. *Water resources research, Vol. 16, No. 4, pp. 638-642*