The Economics of Shallow Lakes: Multiple steady states, strategic interactions and hysteresis.

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Plan of the lecture

- Introduction to shallow lakes
- Introduction to differential games
- Static equilibria in Shallow Lakes model
- Dynamic equilibria in Shallow Lakes model

Description

- Lake is used as a source of water by **several** communities
- This results in loading of a lake with phosphorus (pollution)
- It is also used for recreation (fishing..)
- The cleaner is the lake, the higher utility from it (preservation)
- Optimal choice of loading level by each community independently - **common property equilibrium**
- Joint optimization of welfare - **command optimum equilibrium**

Features

Shallow lakes model has number of interesting features:

- Multiplicity of equilibria
- Optimal management does not help always
- Irreversibility of pollution
- History dependence of optimal policy

We concentrate this time on two first points

Why we need this stuff?

- A clear example of non-linear dynamics
- Non-linearities are everywhere in Economics
- Very typical for environmental problems
- Used frequently as a toy model to verify complicated concepts (and we do)
- Climate change is another type of shallow lakes!

Stylized components

- **Non-linear dynamics** of the state (pollution concentration)
- Several (but finitely many) **interacting agents**
- Finite **absorptive capacity** of a lake
- Intertemporal consequences are essential: pollution builds up gradually
- **Irreversibility** thresholds for pollution
- Separate control for every agent, but single common state (state of the lake).

Game theory recap

- The **game** consists of:
  1. Set of players, N;
  2. Set of strategies of players, $\{s_1, ., s_n, .., s_N\}$
  3. Set of associated payoffs for all players: $V_n(s_1, .., s_N)$ depending on strategy choice of all players
  4. Value of the game V: total payoff for all players under chosen strategic profile.
- Games are in terms of payoff:
  1. Non-cooperative: each player maximizes his/her own payoff;
  2. Cooperative: players maximize joint payoff function
- In terms of dynamics:
  1. Static: only one action per player, maximization of associated pay-off
  2. Dynamic: game played over multiple periods, maximization of total pay-off in time
- Full information, asymmetric information, incomplete information games.

Game-theoretical equilibrium concepts

In cooperative and non-cooperative games different concepts are used:

- Min-Max solution: each player minimizes maximum losses form actions of the other (von Neumann-Morgenstern concept)
- Nash equilibrium (NE): game is in NE, if no player alters his/her strategy given others will not do so
- Shapley value (SV): distribution of total payoff among cooperating players

Example: Prisoner's Dilemma

Dynamic games concepts

Dynamic games types:

- Repetitive games: the same game is played T times, $t \in \{1, 2, 3, .., T\}$
- Differential games: the game is played over real time with strategies being functions of time
- Evolutionary games: Repetitive games with **mutations**

We will work only with differential games.
Dynamic equilibria concepts:

- Open-loop Nash equilibria
- Feedback Nash equilibria
- Repetitive Nash equilibria
- Evolutionary stable equilibria

Open loop and Feedback NE

We first define those:

- **Open loop NE** of a differential game is the set of strategies of players, $\{s_1^O(t), .., s_N^O(t)\}$ such that they maximize total payoffs of all players given **optimal** strategy profile of all other players-
- **Feedback NE** of a differential game is the set of strategies of players $\{s_1^F(t), .., s_N^F(t)\}$ maximizing their payoffs given **any** strategy profile of all other players
- Equilibrium is **subgame perfect** if it is an equilibrium of any subgame of the game.

Time consistency

- Weak time consistency:
  1. Continuation of the optimal trajectory;
  2. Only at the equilibrium;
  3. Along the path optimality (NOT subgame perfect).
- Strong time consistency
  1. Out-of-equilibrium concept;
  2. Anywhere optimality;
  3. IS subgame perfect;
  4. Markov-type consistency.

Differential game

The differential game (DG) consists of:

- Objective functional J_i for every player $i \in N$;
- Dynamic constraints on state variables (common or individual)
- Set of initial/boundary conditions on states/controls

So the DG is like an optimal control problem with several participants!

- Cooperative DG: players maximize $J = \sum_i^N J_i$
- Non-cooperative DG: each player maximize his/her own J_i
- Cooperative DG = Command optimum + solution concept (SV)
- Non-cooperative DG $\neq$ Competitive outcome!

Open loop solution of differential game

- Construct Hamiltonians for all players
- Derive F.O.C.s and co-states for all players
- Assume others' strategies are fixed
- Solve algebraic system on strategies
- Substitute these back into dynamics and solve resulting DE's.

Algorithm is the same as for optimal control except there are many controls to find.

Setup

- There are $n \neq \infty$ communities residing on the lake
- They use lake as a waste sink and for recreation
- Lake has **multiple equilibria**:
  - Eutrophic state: high pollution, low biodiversity
  - Oligotrophic state: low pollution, high biodiversity
- Depending on parameters and initial state both might be optimal
- It is harder to return to clean state from dirty one then vice versa: **irreversibility**

Model

Each community has a payoff function

$$W(t) = \ln(a(t)) - c\,x^2(t) \tag{1}$$

- $\ln a$ is a value of a lake as a waste sink;
- x^2 is a value of ecological services from the lake (water quality);
- Coefficient c measures relative weight of these.

There are n communities (**players**) and thus

$$\sum_{i=1}^{n} \ln a_i - ncx^2 \tag{2}$$

is the objective functional for central planner.

Optimal management

The static central management problem:

$$\max_a \sum_{i=1}^{n} \ln a_i - ncx^2$$

$s.t.$

$$a - bx + \frac{x^2}{x^2+1} = 0, a = \sum_{i=1}^{n} a_i. \tag{3}$$

yields

$$b - \frac{2x}{(x^2+1)^2} - 2cx(bx - \frac{x^2}{x^2+1}) = 0 \tag{4}$$

Properties

- Optimal management does not necessarily lead to a oligotrophic state
- With low c it is optimal to go to eutrophic state
- For $c \leq 0.36$ there is one global maximum for x behind the **flip point**
- For $c = 1$ the only global maximum gives oligotrophic state with $x^* = 0.33$
- This is achieved for the loading level $a^* = 0.1$, while the flip occurs for $a = 0.1021$
- **Edge of hysteresis.**

Common property solution

Non-cooperative communities maximize their payoff which results in the **Nash equilibrium**:

$$\forall i \in [1; n] \subset \mathbb{N} : \max_{a_i} \ln a_i - cx^2$$

$$s.t.$$

$$a - bx + \frac{x^2}{x^2+1} = 0, \; a = \sum_{i=1}^{n} a_i \qquad (5)$$

yielding equilibrium characterization

$$b - \frac{2x}{(x^2+1)^2} - 2\frac{c}{n}x(bx - \frac{x^2}{x^2+1}) = 0 \qquad (6)$$

Properties

- With $n = 2$ there are three equilibria
- One yields oligotrophic state with $x^{N_1} = 0.36$ and $a^{N_1} = 0.1012$, comparable to optimal management
- Another one is in eutrophic state with $x^{N_2} = 1.51$ and $a^{N_2} = 0.2108$
- The third in between is unstable (flip point)
- **Hysteresis effect**: it is harder to go from N_2 to N_1 then vice versa

It is *not* enough to reduce loading back to $a = 0.1$ to obtain the oligotrophic state, but loading have to be reduced till $a^F = 0.0898$.

Setup

- The loading may vary over time, it is the **control**
- Objectives are infinite time horizon functionals:

$$W_i = \int_0^{\infty} \mathrm{e}^{-\rho t}[\ln a_i(t) - cx^2(t)]dt, i = 1, ..., n. \qquad (7)$$

- The phosphorus concentration is the **dynamic constraint**:

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{x^2(t)+1}. \qquad (8)$$

Optimal management

Under optimal management the sum of objectives W_i is maximized:

$$\max_{a_i(\bullet)} \sum_{i=1}^{n} W_i = \max_{a_i(\bullet)} \sum_{i=1}^{n} \int_0^\infty \mathrm{e}^{-\rho t}[\ln a_i(t) - cnx^2(t)]dt \qquad (9)$$

resulting in the evolution of total loading:

$$\dot{a}(t) = -\left((b+\rho) - \frac{2x(t)}{(x^2(t)+1)^2}\right) a(t) + 2cx(t)a^2(t) \qquad (10)$$

Dynamics and steady states under optimal management

- Steady state of a depends on discount rate ρ
- With ρ low enough (long time horizon and planning) there is only one oligotrophic steady state
- As discount rate increases 3 SS appear
- With $\rho = 0.03$ the solution converges to the static model
- This unique steady state is saddle-type stable.

Common property solution

- Each community maximizes its own functional

$$\max_{a_i(\bullet)} W_i = \max_{a_i(\bullet)} \int_0^{\infty} \mathrm{e}^{-\rho t}[\ln a_i(t) - cx^2(t)]dt \qquad (11)$$

- subject to the common dynamics of the lake
- This results in the loading dynamics:

$$\dot{a}(t) = -\left((b+\rho) - \frac{2x(t)}{(x^2(t)+1)^2}\right) a(t) + 2\frac{c}{n}x(t)a^2(t) \qquad (12)$$

Equilibria properties

- Under suitable values $n = 2, b = 0.6, c = 1, \rho = 0.03$ three equilibria exist
- The middle one is unstable and two others are stable
- They correspond to oligoptrophic and eutrophic states respectively
- The **Skiba point** separating them exists
- The **hysteresis effect** is also observed.

Conclusions

- Game theory: Nash equilibrium concept
- Subgame-perfect NE and time consistency
- Shallow lakes: a workhorse of many dynamic phenomena
- We observed multiple equilibria both in planner and common property cases
- This is necessary for history dependence

Next lecture

- We concentrate on irreversibility and history dependence
- Bifurcations are introduced
- We use the same shallow lakes model to obtain global dynamics
- Feedback NE in this game and economic policy design
- (Additional) Paper: Kiseleva T., Wagener F. (2011) Bifurcations of optimal vector fields in the shallow lake model. *Journal of Economic Dynamics and Control 34(5)*, pp. 825-843

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