The Economics of Shallow Lakes: Irreversibility and history dependence.

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Plan of the lecture

- Reminder of the last lecture
- Hysteresis: irreversibility of pollution
- Skiba points: history dependence of equilibria
- Global dynamics of Shallow Lakes
- Economic policy in Shallow Lakes

Setup

- Lake is used as a source of water by **several** communities;
- This results in loading of a lake with phosphorus;
- This loading determines the final steady state of the lake: dirty or clean;
- The existence of hysteresis effect depends on the **sedimentation** rate;
- A lot of natural resource systems display this type of dynamics;
- Depending on parameters the system may have 1 or 3 steady states.

Dynamics

- The state of the lake is described by the amount of phosphorus in the water:

$$\dot{P}(t) = L(t) - sP(t) + r\frac{P^2(t)}{P^2(t) + m^2},\ P(0) = P_0 \qquad (1)$$

- Normalized equation we work on is:

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{x^2(t) + 1}, x(0) = x_0. \qquad (2)$$

Types of solutions

Shallow lakes model has:

- Static command optimum
- Static 'competitive' equilibria
- Dynamic command optimum
- Dynamic common property equilibria.

We focus only on dynamic ones

Derivation for command optimum

The control problem yields:

1. F.O.C.s for each i:

$$\frac{1}{a_i(t)} + \lambda(t) = 0 \tag{3}$$

2. Single co-state equation:

$$\dot{\lambda}(t) = \left[(b+\rho) - \frac{2x(t)}{(x(t)^2+1)^2}\right]\lambda(t) + 2ncx(t) \tag{4}$$

3. Expressing a_i through λ yields

$$\dot{a}_i(t) = \dot{\lambda}(t)a_i^2 \tag{5}$$

4. Using symmetry $\forall i,j \in n: \ a_i = a_j$ and multiplying by n we get

$$\dot{a}(t) \overset{\lambda(t)=-\frac{n}{a(t)}}{=} -\left[(b+\rho) - \frac{2x(t)}{(x(t)^2+1)^2}\right]a(t) + 2cx(t)a(t)^2 \tag{6}$$

Derivation for OLNE

1. F.O.C.s for each i:
$$\frac{1}{a_i(t)} + \lambda_i(t) = 0 \tag{7}$$
2. Co-state equation for each i:
$$\dot{\lambda}_i(t) = \left[(b+\rho) - \frac{2x(t)}{(x(t)^2+1)^2}\right] \lambda_i(t) + 2cx(t) \tag{8}$$
3. Expressing a_i through λ_i yields
$$\dot{a}_i(t) = \dot{\lambda}_i(t) a_i^2 \tag{9}$$
4. Using symmetry $\forall i,j \in n: \ a_i = a_j$ and multiplying by n we get
$$\dot{a}(t) \overset{\lambda_i(t) = -\frac{1}{a_i(t)}}{=} - \left[(b+\rho) - \frac{2x(t)}{(x(t)^2+1)^2}\right] a(t) + 2c/nx(t)a(t)^2 \tag{10}$$

Multiplicity of steady states

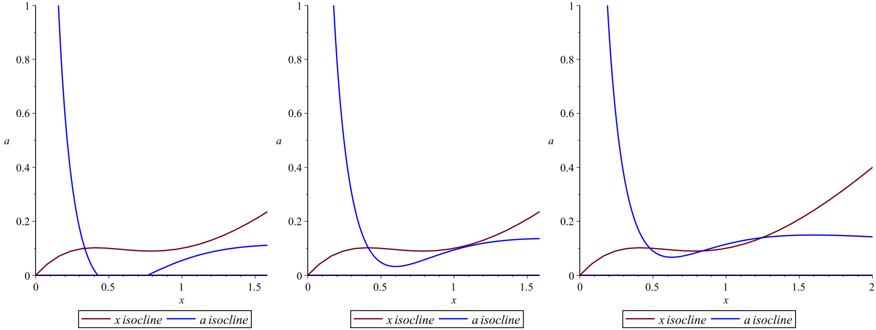
- The steady states of a command system are thus defined by isoclines:

$$\bar{a} - b\bar{x} + \frac{\bar{x}^2}{\bar{x}^2 + 1} = 0$$

$$-\left[(b+\rho) - \frac{2\bar{x}}{(\bar{x}^2+1)^2}\right]\bar{a} + 2c\bar{x}\bar{a}^2 = 0 \qquad (11)$$

- Their intersections are steady states, depending on 3 parameters, b, c, ρ
- Variation in b affects both isoclines, but c, ρ only the a isocline.

Illustration: variation in ρ

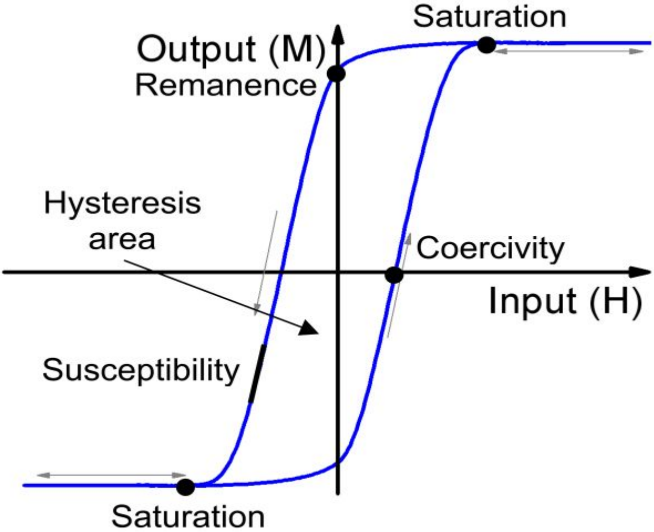


0.01 0.088 0.13

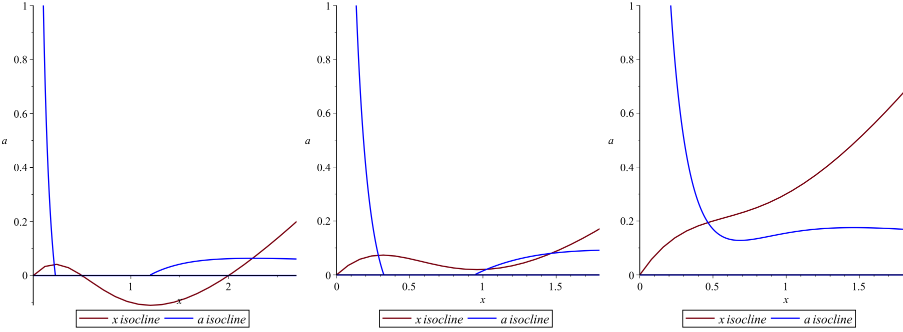
Hysteresis effect

- Natural phenomenon
- Dynamical systems property
- Irreversibility is one particular example
- Hysteresis can be:
  - Reversible
  - Irreversible
- Irreversible: increase a a bit and never comes back
- Reversible: decrease a much more to come back, than increasing it

Illustration



Variation in b



0.4 0.52 0.8

Implications

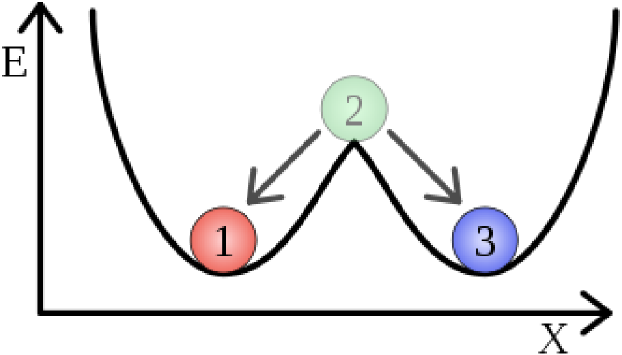
- If irreversible hysteresis exists ($b < 0.5$), only **one** equilibrium is chosen;
- **Edge of hysteresis**: optimal management is not **robust**;
- Hysteresis thresholds: once loading exceeds given level, there is no way back
- Policy implications...

Path dependency

- Given a dynamical system $\dot{x} = f(x)$ path dependency is:
  1. For different x_0 system ends up in different equilibria;
  2. There exist **thresholds**, separating basins of attraction of different equilibria;
  3. Slight change in initial conditions cause convergence to different states.
- Multiplicity of equilibria is necessary
- Is frequent for **bistable** systems:
  1. The system has 3 steady states;
  2. They are ordered as

$$\bar{x}_1 < \bar{x}_2 < \bar{x}_3 \tag{12}$$

  3. The medium steady state is unstable
  4. $\bar{x}_{1,2}$ are (saddle) stable.



- Positions $1, 3$ are stable, position 2 is unstable.
- The system is **bistable**.

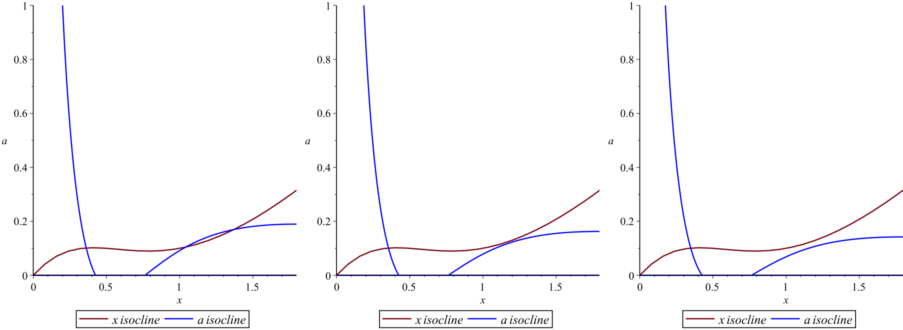
Shallow lakes path dependency

- Once the system has 3 equilibria it is bistable;
- Is this a generic outcome?
- Can we switch the system to desired state and how much does it cost?
- Will the convergence to desired state be short or long?
- **Answer**: Bifurcations theory and Skiba points.

Incursion into bifurcations theory

- **Bifurcation** is qualitative change of dynamical system with parameters' changes
- Co-dimension 1 bifurcations: only one parameter is changed
- Co-dimension 2,3 bifurcations: 2 or 3 parameters change simultaneously
- Global bifurcations: heteroclinic connections
- Local bifurcations: saddle-node bifurcation
- Bifurcation diagram: plots number of steady states changes w. r. t. parameter changes
- Steady state is **structurally stable**, if it does not appear/disappear with parameter changes.

Illustration: saddle-node bifurcation in c



0.6 0.7 0.8

Remarks

- In this example only clean state is structurally stable in c, since it does not disappear
- Saddle-node bifurcation: saddle-stable and unstable states collide at $c = 0.7$ and disappear afterwards
- There are many other types of bifurcations, but all follow the same picture
- In shallow lakes saddle-node is the **only one** local bifurcation possible
- Still, heteroclinic connections are also possible

Heteroclinic connections

- Every steady state has **stable** and **unstable** manifolds
- Stable manifold is the set of paths leading into the steady sate
- Unstable manifolds is the set of paths leading out of the steady state
- Unstable steady state has only unstable manifold and stable steady state has only stable one;
- **Saddle**-stable state has both!
- **Heteroclinic connection** is the path, connecting unstable manifold of one saddle with the stable manifold of the other

Illustration

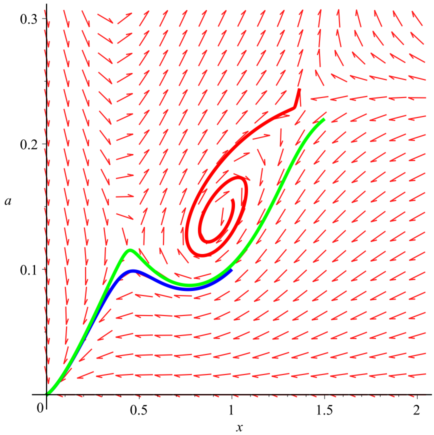


Figure: Three trajectories with different x_0

Skiba point

- The last figure illustrates the **Skiba point**
- With $x_0 = 1$ there are two different trajectories, leading to clean and dirty states
- Once $x_0 < 1$, all trajectories go to clean one, and vice versa;
- However, there is a heteroclinic connection from dirty to clean one: **hysteresis**
- Once there are no such connections **and** Skiba point exists:
  1. There is perfect separation of outcomes;
  2. Once initial pollution is lower than x_0^S, all optimal paths grant clean state;
  3. Once initial pollution exceed this level, the only optimal outcome is the dirty one.

Since x_0 is **history**, this is history dependence.

Overview

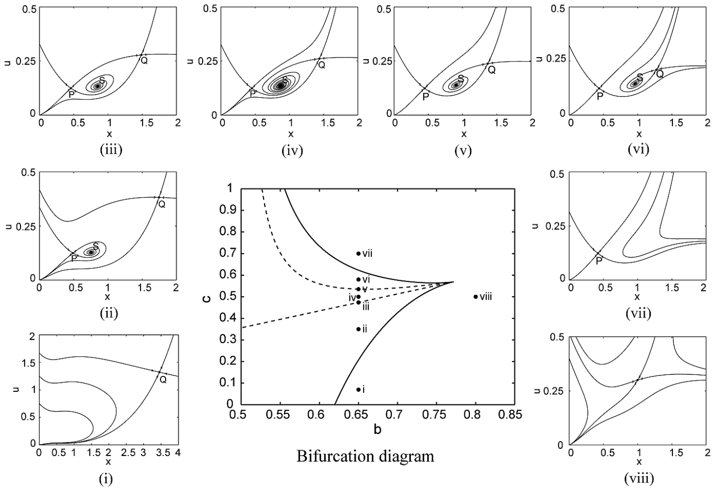
- Shallow lakes may have up to 3 equilibria;
- These are saddle-stable and one unstable;
- Skiba point might exist in case of three steady states;
- Once heteroclinic connections exist, global dynamics is important.

What types of dynamics may be observed?

Types of dynamics

- Equilibria:
  1. Unique equilibrium: either dirty or clean, no policy relevant
  2. Two equilibria, one saddle, one unstable: only stable realizes;
  3. Three equilibria: history dependence.
- Convergence:
  1. No heteroclinic connections: all $x_0 < x_0^S$ converge to clean state, all positions with $x_0 > x_0^S$ converge to the dirty one;
  2. There is a heteroclinic connection: only one equilibrium realizes in the **long-run**, but speed of convergence is different!

Global picture (from Wagener (2003))



Problem

- Every community maximizes:

$$W_i = \int_0^\infty \mathrm{e}^{-\rho t} \left[ln a_i(t) - \tau(t) a_i(t) - c x^2(t) \right] dt \qquad (13)$$

- Optimal loading is thus:

$$\frac{1}{a_i(t)} - \tau(t) + \lambda_i(t) \qquad (14)$$

- The corrective tax is difference in shadow costs:

$$\tau(t) = -\lambda(t) + \lambda_i(t) \qquad (15)$$

- Which is time-dependent

Distortionary taxation

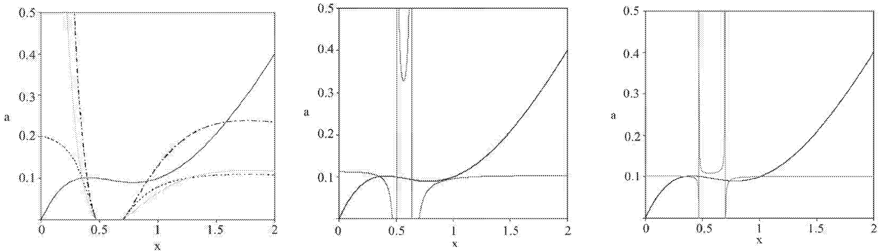
- The **constant** tax is defined such as to switch the game to command optimum:

$$\tau^* = -\frac{(n-1)\lambda^*}{n} = \frac{n-1}{a^*} \tag{16}$$

- This tax will change the individually optimal loading scheme:

$$\dot{a}(t) = -\left[(b+\rho) - \frac{2x(t)}{(x(t)^2+1)^2}\right]\left[a(t) - \frac{\tau^*}{n}a^2(t)\right] + 2c/nx(t)a(t)^2 \tag{17}$$

Illustration of dynamics



Role of interactions

- Increase in n leads to more differences in dynamics;
- There might exist several steady-state loading levels under taxation
- With $n \to \infty$ it simply does not work.
- Why?

Role of interactions

- Increase in n leads to more differences in dynamics;
- There might exist several steady-state loading levels under taxation
- With $n \to \infty$ it simply does not work.
- Why?

- The loading level depends on n;
- Convergence to a^* is not uniform (recall heteroclinic connections)
- Policy works well only for unique steady state and symmetric communities
- Environment is **more** complex than the regulation!

Implications for policy design

- The economic policy should be relevant to the model
- I. e. the more complex is the environment the more complex should be the policy
- Impulse control policy may be relevant:
  - Short-term policy impulses implemented to switch the system to convergence to unique steady state
  - Plus constant tax
- Issue of robustness: precision of the policy is opposite to its robustness to initial values

Take home things

- Many economic phenomena are highly non-linear;
- This requires global dynamic analysis
- Multiplicity of equilibria and history dependence issues
- Strategic interactions: common property equilibrium complexity grows in n
- Bifurcations and Skiba points
- State-dependant vs. simple policy designs

Next lecture

- We switch to Macro-perspective;
- Optimal management of resource-dependent economy
- Can technology substitute for the resource?
- **Paper**: Dasgupta P. and G. heal (1974) The Optimal Depletion of Exhaustible Resources. *The Review of Economic Studies, 41 (SI)*: pp. 3-28.