Optimal resources usage: Neoclassical growth theory and resources

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Plan of the lecture

Motivation: cake-eating problem

Model with production

CES technology and isoelastic utility

Technological breakthrough

Concluding remarks

Hotelling's generalization

- We start with the simplest setup
- No production, just consumption of the resource stock, S
- Generalization: there is a substitute for the resource, M
- We then define the **sustainable** extraction path

Formal setup

- Social planner maximizes utility:

$$\max_{C} \int_{0}^{T} \mathrm{e}^{-\delta t} U(C(t)) dt \tag{1}$$

- Given the source of consumption is the resource stock S:

$$\dot{S} = M - C(t) \tag{2}$$

- **Important**: M is constant

Augmented Hamiltonian

- The problem resembles Hotelling's, but with multiple phases
- Phases follow from **augmented Hamiltonian**:

$$\mathcal{L} = \mathcal{H} + q(t)S(t) =$$
$$U(C(t)) + p(t)(M - C(t)) + q(t)S(t) \quad (3)$$

with $q(t) \geq 0, q(t)S(t) = 0$ - **complementary slackness condition**

Modified Hotelling

- F.O.C. implies Hotelling's rule:

$$p(t) = U'(C(t)) \tag{4}$$

- Co-state equation is

$$\dot{p} = -q(t) + \delta p(t) \tag{5}$$

- Defining **elasticity of marginal utility**

$$\eta(C) = -CU''(C)/U'(C) \tag{6}$$

- We arrive to modified Hotelling's rule:

$$\frac{\dot{C}}{C} = -\frac{\delta}{\eta(C)} + \frac{q(t)}{\eta(C)U'(C)} : \tag{7}$$

Multiple phases of extraction

- Multiple extraction phases as for renewable resource:
  - As long as $S(t) > 0$ we have $q(t) = 0$ and hence

$$C^A : \frac{\dot{C}}{C} = -\frac{\delta}{\eta(C)} < 0 \tag{8}$$

  - As soon as $S(t) = 0$ we have $q(t) > 0$ and extraction is given by (7):

$$C^B : -\frac{\delta}{\eta(C)} + \frac{q(t)}{\eta(C)U'(C)} = 0 \tag{9}$$

  - It follows that

$$q(t > T) = \delta U'(M);\ C(t > T) = M \tag{10}$$

Determination of T

- Time-to-extraction T is characterized by:

$$\int_0^T C(t)dt = S(0) + MT \tag{11}$$

- To obtain T one needs:
  1. Solve for dynamics of $C(t < T)$ via (8):

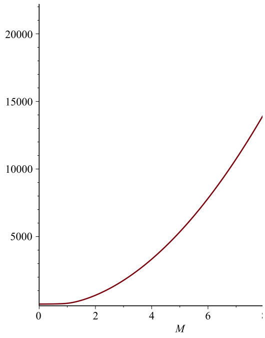
$$C(t < T) \overset{U(C)=C^\alpha}{=} \sqrt{-2\,\alpha\,\delta\,t + C0^2 + 2\,\delta\,t} \tag{12}$$

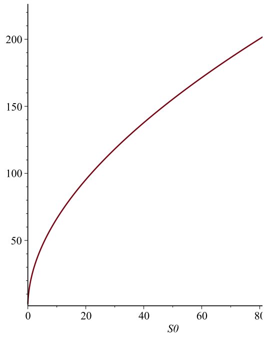
  2. Insert this into the integral:

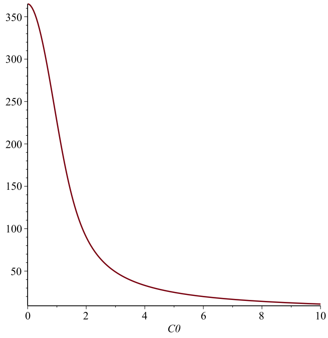
$$1/3\,\frac{C0^3}{\delta\,(\alpha-1)} - 1/3\,\frac{\left(-2\,T\alpha\,\delta + C0^2 + 2\,T\delta\right)^{3/2}}{\delta\,(\alpha-1)} = MT + S0 \tag{13}$$

  3. Solve resulting equation on T

Illustration: influence of parameters on T







Setup

- We add (neoclassical) production into the model:

$$F(K,R):\ F(0,R)\geq 0, \lim_{K\to\infty}\frac{\partial F}{\partial K}<\delta, \lim_{K\to 0}\frac{\partial F}{\partial K}>\delta \qquad (14)$$

- **Capital** accumulates through savings:

$$\dot{K}=F(K,R)-C \qquad (15)$$

- Resource R is extracted for production purposes **with no regeneration**:

$$\dot{S}=-R, \qquad (16)$$

- Planner is maximizing utility from consumption:

$$\max_{C,R}\int_0^\infty \mathrm{e}^{-\delta t}U(C(t))dt \qquad (17)$$

Optimal policy

- The augmented Hamiltonian is:

$$\mathcal{L} = e^{-\delta t}U(C) + e^{-\delta t}p(t)(F(K,R) - C) - \lambda(t)R + e^{-\delta t}\mu(t)R \tag{18}$$

- There are two F.O.C.s:

$$p(t) = U'(C)$$
$$\lambda(t) = e^{-\delta t}\left(\mu(t) + p(t)\frac{\partial F}{\partial R}\right) \tag{19}$$

- This is another form of the **Hotelling's rule**

Comment on Hotelling's rule

- Hotelling's rule says how the price of the resource should be optimally set
- As long as $R(t) > 0$ (there is resource left), we have $\mu(t) = 0$ and

$$\lambda = \mathrm{e}^{-\delta t} p(t) \frac{\partial F}{\partial R} \tag{20}$$

- **Relative** (shadow) price of the resource, $\lambda / p(t)$ equals its (discounted) marginal product
- This holds as long as extraction costs are neglected
- **NB**: Resource price never equals its marginal extraction costs!

Time to depletion

- Whether or not the resource will be depleted in finite time, depends on its **productivity**:
  - If marginal product of the resource is unbounded, its exploitation is infinite:

$$\lim_{R\to 0} \frac{\partial F}{\partial R} = \infty \to R > 0 \forall t \geq 0. \tag{21}$$

  - Time of exploitation is finite, if resource is inessential and its average product is bounded ($x = K/R$):

$$\lim_{x\to\infty} (f(x) - xf'(x)) = \gamma < \infty, F(K, 0) > 0 \tag{22}$$

    then

$$R(t) \geq 0 : \; T \geq t > 0 \, ; R(t) = 0 : \; t \geq T \neq \infty. \tag{23}$$

Consequences

- In this general setting there is **resource trap**
- If resource is essential, it cannot be fully depleted
- Thus consumption decreases over time to zero
- Capital cannot substitute for resource (Cobb-Douglas)
- Absence of technology prevents further growth
- If resource is inessential, it is fully exploited at $t = 0$.

Specification

To obtain steady states and growth paths we specify the economy as:

- CES production technology:

$$F(K,R) = \left[\beta K^{(\sigma-1)/\sigma} + (1-\beta) R^{(\sigma-1)/\sigma}\right]^{\sigma/(\sigma-1)} \tag{24}$$

- With $\sigma \leq 1$ resource is essential (Cobb-Douglas with $\sigma = 1$)
- With $\sigma > 1$ it is inessential
- Isoelastic utility function: $\eta(C) = \eta > 0$

Steady state of the CES economy

With $\rho = \lim_{K/R \to \infty} f(K/R)/(K/R)$:

- Economy growth rate is

$$\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\rho - \delta}{\eta} \tag{25}$$

- Resource exploitation growth rate:

$$\frac{\dot{R}}{R} = \frac{\rho - \delta}{\eta} - \sigma\rho < 0 \tag{26}$$

- Consumption-capital ratio:

$$\frac{\bar{C}}{\bar{K}} = \rho + \frac{\delta - \rho}{\eta} \tag{27}$$

For $\sigma \leq 1$ we have $\rho = 0$ thus growth is **negative**

Case $1 > \sigma > 0$

- Total output is bounded by resource if $\sigma < 1$
- The relative resource price $\lambda/p(t)$ is monotonically increasing
- The shadow price tends to

$$(1-\rho)^{\sigma/(\sigma-1)} \tag{28}$$

- The consumption profile tends to zero:

$$\lim_{t\to\infty} C(t) \to 0 \tag{29}$$

since more resource has to be substituted by capital stock

Cobb-Douglas case

We now assume $x = K/R, f(x) = x^{\alpha}$, and $\sigma = 1$.

- The steady state capital to resource ratio is:

$$\bar{x} = \left((1-\alpha)t + x_0^{1-\alpha}\right)^{1/(1-\alpha)} \tag{30}$$

- The relative price of the resource is then

$$\lambda/p(t) = (1-\alpha)^{1/(1-\alpha)} t^{\alpha/(1-\alpha)} \tag{31}$$

- The optimal consumption profile is

$$\bar{C}(t) = \bar{C}(0)\bar{x}(0)^{\alpha/\eta}\left[(1-\alpha)t + x_0^{1-\alpha}\right]^{\alpha/\eta(1-\alpha)} \mathrm{e}^{-(\delta/\eta)t} \tag{32}$$

- Resource extraction is

$$\dot{R} = \bar{C}(0)\bar{x}(0)^{\alpha/\eta}\left[(1-\alpha)t + x_0^{1-\alpha}\right]^{\alpha/\eta} \mathrm{e}^{-(\delta/\eta)t} \tag{33}$$

Illustration: Consumption profile

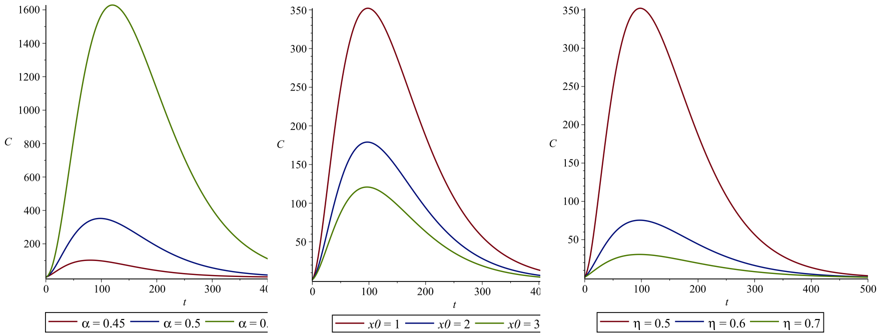
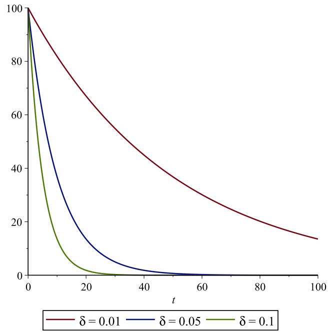
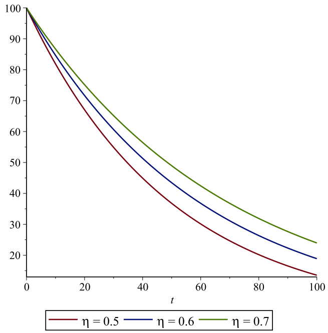


Illustration: Resource profile





Single-peaked consumption profile

- Once technology is Cobb-Douglas, there is a single max of consumption
- This had consequences to the theory:
  1. Environmental Kuznets curve (EKC)(fails empirically)
  2. Endogenous growth theory
  3. Renewable energy studies
- Overall, Environmental Economics studies have originated from this fact
- Discount rate is of vital importance - intergenerational equity question

Technology

- Technology may turn the essential resource into inessential
- Suppose at time T some (renewable) ultimate energy source appears
- Then we have **piecewise-defined** problem:
  1. Optimize resource use up to T
  2. Switch to optimal usage of renewable resource after T
  3. determine T (unknown)

Breakthrough

- Discovery of a perfectly durable commodity
- It yields a constant flow of 'service' M
- Production function changes: $F(K(t), R(t) \rightarrow G(K(t), Z(t))$
- G is new technology, Z is utilization of a service
- It is perfect substitute for the resource

After breakthrough

After the breakthrough occurs, the problem is to maximize utility subject to:

$$\begin{aligned}
&\forall t \in [T, \infty): \\
&\dot{K} = G(K(t), Z(t)) - C(t), \\
&\dot{V} = M - Z(t), \\
&V(T) := S_0 - \int_0^T R(t)dt
\end{aligned} \tag{34}$$

With $G(K, Z)$ being the new production technology, Z being renewable resource

- K_T is given by before breakthrough problem;
- T is either random, or endogenously defined
- $V(T)$ is **initial condition** on remaining resource

Before the breakthrough

Before the breakthrough we utilize the exhaustible resource:

$$\dot{K} = F(K, R) - C$$
$$\dot{S} = -R \tag{35}$$

which is pursued **until the substitute is discovered**

- At any point in time there is a probability of breakthrough
- It is certain that at some point it will happen:

$$\omega(t) > 0, \int_0^\infty \omega(t)dt = 1 \tag{36}$$

Uncertainty structure

- At each $t < T$ there is a non-zero probability of a breakthrough
- The $\omega(t)$ is **probability density**
- The actual probability that breakthrough happens after any t is

$$\Omega(t) = \int_t^\infty \omega(\tau) d\tau \tag{37}$$

- At any time we weight probability that breakthrough have not happened yet ($\Omega(t)$) and that it will happen exactly at t, $\omega(t)$
- Yields a certainty equivalence problem

Certainty equivalent problem

- Denote $W(K(t), V(t))$ maximized value of the after-breakthrough problem for $T = t$
- Then before breakthrough planner is maximizing

$$\mathbb{E}\int_0^\infty \mathrm{e}^{-\delta t} U(C)dt = \int_0^\infty \mathrm{e}^{-\delta t} U(C)\Omega(t) + \omega(t)W(K(t), V(t))dt \tag{38}$$

- Subject to constraints

$$\begin{aligned} \dot{K} &= F(K, R) - C, \\ \dot{S} &= -R, \\ S(t) &= V(t) \end{aligned} \tag{39}$$

- The last implies the **same** resource stock is used by breakthrough technology

Conditional evolution of the economy

Denote $\Psi(t) = \omega(t)/\Omega(t)$: likelihood of breakthrough at t;

- Then consumption path depends on this likelihood:

$$\frac{\dot{C}}{C} = \frac{F_K - \delta + \Psi(t)(W_K - U'(C))/U'(C)}{\eta(C)} \tag{40}$$

- After the breakthrough we get standard optimal problem with renewable resource
- Outcome depends on probability density, $\omega(t)$!

Assumptions

- Once breakthrough occurs, old capital **has no value**:

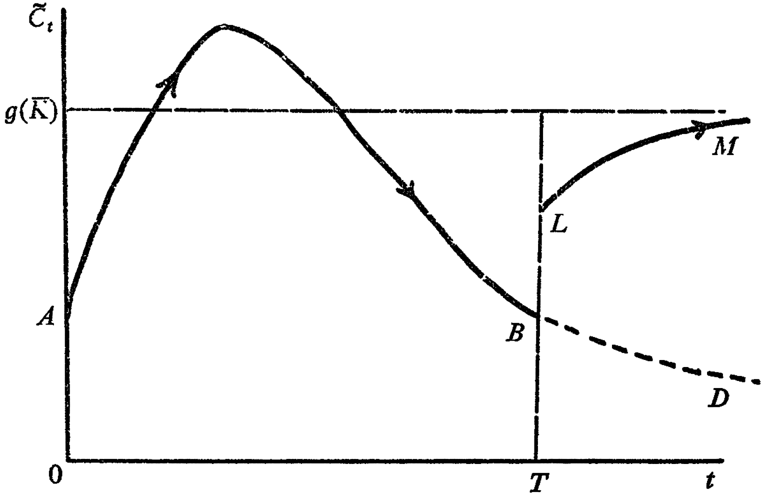
$$W_K = W_V = 0 \tag{41}$$

- Redefining $x = K/R$ as before we get

$$\frac{\dot{C}}{C} = \frac{f'(x) - (\delta + \Psi)}{\eta(C)},$$
$$\dot{x} = \sigma f(x) \tag{42}$$

- So consumption profile before breakthrough is the same as without it (corrected for Ψ)

Consumption profile



Comments

- With (many) simplifying assumptions it turns out that:
  1. Potential breakthrough increases discount rate by $\Psi(t)$
  2. Thus it is optimal to consume more (extract resource faster)
  3. At the breakthrough there is a jump: economic shock
- This solution is non-realistic:
  1. No costs of research
  2. Breakthrough comes with certainty and for free
  3. Old exhaustible resource transforms into renewable, but with different technology.

Other ways out

- Endogenous growth: replace resource with R&D (controlled one)
- Rate of resource discovery: S may grow with time as a consequence of technology
- Human capital not depending on resource

Take home message

- Essential vs inessential resources
- In the economy with essential exhaustible resource only technology may grant ongoing growth
- In renewable world: cake-eating with switching at T
- In non-renewable world: intergenerational equity and investments rule

Conclusion

- Neoclassical thought generates pessimistic predictions over resources
- Thus, non-formal arguments have been used
- The problem impacted trade theory and intergenerational justice theory
- Today reconsidered in endogenous growth
- Conclusion still pretty much holds
- Crucial role of empirical validation: substitution elasticity

Next lecture (tentative)

- Intergenerational equity
- Could we grant non-decreasing consumption to future generations
- Given exhaustible resource is present?
- Counterargument by Solow: better invest it all now
- Paper: Solow (1974) Intergenerational equity and exhaustible resources. *The Review of Economic Studies, 41*, pp. 29-45 (the same issue on Symposium as today's paper)