# Introduction to growth and technical change

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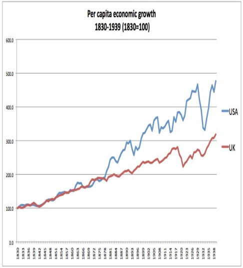
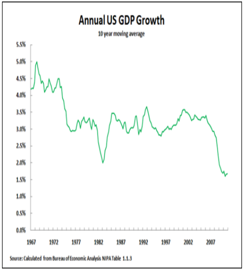
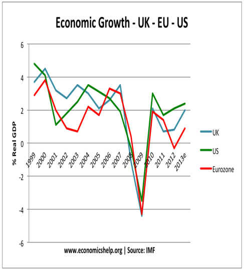
# What is the growth theory?

- Collection of models trying to explain economic growth
- Conceptual framework for describing modern economies
- Normative framework telling how to maximize growth rates (and welfare)

# Facts about growth



# Motivation

- Why do we need all this stuff?
- Why do we need to construct new growth models?
- Interaction with real world and empirical research
- Policy implications

# What is the dynamical system?

- Includes time as an independent variable
- Tracks changes of some function over time;
- Includes one or more motion laws
- Can be discrete, continuous, hybrid, delayed...
- Describes numerous natural and social phenomena in time

$$\frac{d}{dt}x := \dot{x} = f(x, t)$$

- **Solution** of dynamical system is an explicit function of time (**evolution function**)

$$f(x, t) \mapsto x = \Phi(t)$$

# Classification

- Dynamical system is **autonomous** if $f(x, t) = f(x)$, the equation does not explicitly depend on time
- Dynamical system is **linear**, if $f(t, x) = A(t)x + B(t)$, if it is linear in the state variable $(x)$
- Dynamical system is **finite-dimensional**, if $\dim x < \infty$
- Dynamic system is an ODE system, if its solution is function of time only $x(t) = \Phi(t)$

# Existence and uniqueness

- Existence and uniqueness of solutions for any ODE are not granted
- Even when the solution exists it is far more often than not impossible to obtain
- It is important to ensure that the solution exists to make any qualitative conclusions about the dynamical system
- This is not the case for discrete-time systems.

# Fixed points and steady states

- A set of fixed points (steady states) of an ODE system is defined by $\dot{x} = 0$
- It is possible to classify fixed points according to their stability
- The higher the dimensionality of the system, the more complicated is the classification
- The stability analysis is based on the stability of linear systems and the **Jacobian matrix**.

# Linear autonomous systems

Let the dynamical system be given by

$$\dot{x} = Ax;$$
$$x \in \mathbb{R}^n.$$

Then the only fixed point (and solution) of the system is $\bar{x} = 0$.

- If all algorithm eigenvalues of $A$ have negative real parts, then every solution is stable ($\bar{x}$ is a sink)
- If any eigenvalue of $A$ have positive real part, then every solution is unstable ($\bar{x}$ is a source)
- If some of the eigenvalues of $A$ have zero real parts and all other have negative real parts, then let
  $\lambda = i\sigma_1, i\sigma_2, .., i\sigma_m$
  be eigenvalues with zero real parts. If the multiplicity of all such eigenvalues is one, then every solution is stable.

# Classification of stability regimes for 2-dim systems

| Type | $Re(\lambda)$ | $Im(\lambda)$ |
|---|---|---|
| Source node | $> 0$ | $= 0$ |
| Sink node | $< 0$ | $= 0$ |
| Saddle | $\lambda_1 > 0, \lambda_2 < 0$ | $= 0$ |
| Center | $= 0$ | $\neq 0$ |
| Spiral source | $> 0$ | $\neq 0$ |
| Spiral sink | $< 0$ | $\neq 0$ |

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# How to study stability

1. System is **linear**:
   - Take system matrix $A$ and calculate its eigenvalues
   - Stability given by signs of $\lambda$
2. System is **non-linear**:
   - Calculate the Jacobian $J$ with elements $\{\frac{\partial \dot{x}}{\partial x}\}$
   - Calculate eigenvalues of this Jacobian
   - Evaluate eigenvalues at the steady state, $x = \bar{x}$
   - Signs of eigenvalues of $J$ at $\bar{x}$ provides stability type of **this** steady state (not everywhere).

# Assumptions

- There exists an aggregate production function
- It has the Cobb-Douglas form
- There are two inputs: labour $L$ and capital $K$
- Production function is **static**
- Production function is linearly homogeneous
- Constant returns to scale -no scarce resources
- There is only one commodity - final output $Y$
- Part of this (instant) output is consumed, $C$, and the rest is saved $S$
- The fraction of output being saved is **constant**, $S = sY$
- Labour grows with constant rate: $\dot{L}(t)/L(t) = n$.

# Basic Model

The model is given by only one differential equation:

$$Y(t) = F(K(t), L(t));$$

$$\begin{aligned} \dot{L}(t) &= nL(t); \\ L(t) &= L_0 \mathbf{e}^{nt}; \\ \dot{K}(t) &= sF(K(t), L(t)). \end{aligned}$$

which is a DE in one unknown, since $L(t)$ is known.

# Equation of motion

Capital accumulation equation may be rewritten in intensive form derivation:

$$\dot{k}(t) = sF(k(t), 1) - (n + \delta)k(t) \quad (1)$$

and this equation describes the evolution of the whole economy.

- If we assume Cobb-Douglas production:

$$F(K, L) = K^{\alpha} L^{1-\alpha}$$

- The steady state of this economy is given by the **golden rule** derivation:

$$\hat{k} = \left( \frac{s}{n + \delta} \right)^{1/(1-\alpha)} \quad (2)$$

# Golden rule savings

- Capital $\hat{k}$ depends on the choice of savings
- These are chosen as to maximize consumption:

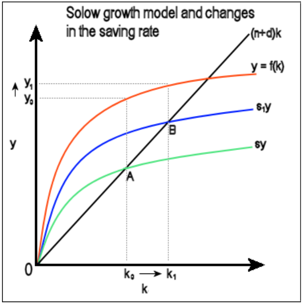
$$C/L = c = (1-s)k^{\alpha} \rightarrow \max$$

- Giving the unique golden rule savings:

$$\hat{s} = \alpha \tag{3}$$

- The steady state is always stable if $\alpha < 1$ Why?

# Illustration



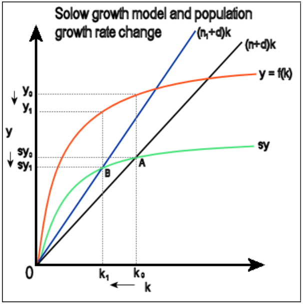




Figure: Parameters in Basic Solow Model

# Concept of technical change and dynamic production

- In the basic model the growth of consumption per capita, $g_c = \dot{c}/c$, after reaching the steady-state, $\hat{k}$ is not possible
- Some source of growth have to be included into the model
- It has been theoretized, that this source is **technical change**.

Solow introduces **dynamic production function**:

$$Y = Y(K, L, t), Y_t > 0. \tag{4}$$

The $t$ represent time and such a form allows for any kind of shifts in production function.
These shifts, however, are by definition, exogenous.

# Neutrality of Exogenous Technical Change

By allowing for specific variable $A$ to reflect technical change, one has more freedom:

- This may be constant or time-varying
- It may be exogenous or endogenous for the model.

Hicks-neutral:

$$Y = Y(K, A(t)L)$$

Solow-neutral:

$$Y = Y(A(t)K, L)$$

Harrod-neutral:

$$Y = A(t)Y(K, L)$$

# Technology in Solow model

Classic Solow model is described with Hicks-neutral technology and Cobb-Douglas production:

$$Y = (AL)^{1-\alpha} K^{\alpha}$$

In this way the population $L$ is replaced with "effective" population $AL$.

Technical change increases labour productivity at constant rate:

$$\dot{A}(t)/A(t) = g_A, A(t) = A_0 \mathbf{e}^{g_A t}.$$

And the effective labour growth rate is thus:

$$\dot{\left(A(t)L(t)\right)}/A(t)L(t) = n + g_A. \tag{5}$$

The equation of motion is obtained in the same way as for the basic model:

$$\dot{k}(t) = sF(k(t), 1) - (n + \delta + g)k(t);$$

$$\hat{\hat{k}} = \left(\frac{s}{n + \delta + g_A}\right)^{1/(1-\alpha)} \tag{6}$$

The consumption per capita is now growing with constant rate $\dot{c}/c = g_A$.

# Role of technical change

- In Solow-Swan model technical change is **factor-enhancing**
- Without technical progress only **extensive** growth is feasible: $\dot{C} > 0$ but $\dot{c} = 0$
- With technical change **intensive** growth equals technical change rate, $g_A$
- This technical progress is **exogenous**: no mechanism suggested behind $A$

# Balanced growth paths

- The **steady-state** is an equilibrium $\dot{k} = 0$
- The balanced growth path of the economy:

$$g \overset{def}{=} \dot{C}/C = \dot{Y}/Y = \dot{K}/K = n + g_A \tag{7}$$

  defines equal growth of all variables
- Thus consumption grows from population $n$ and technology $g_A$
- The balanced growth path (7) implies **exponential growth**
- This is VERY fast, implying turnpike property

# Illustration: Turnpike property

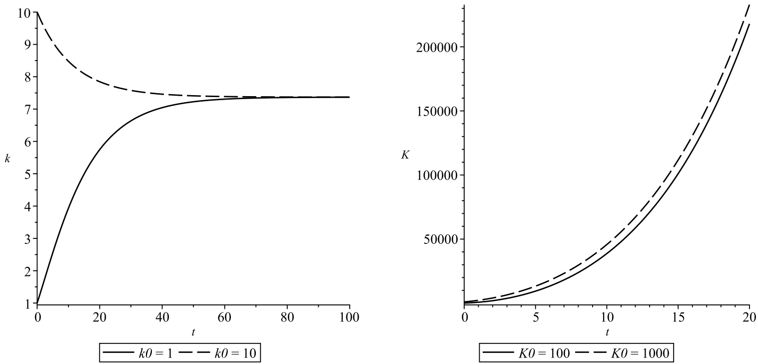



Figure: Exponential growth and steady state

# Intergenerational equity

- Every generation must have the consumption possibilities (per capita) not less than all preceding ones,

$$\forall t > 0 : \; c_t \geq c_{t-1}$$

yielding

$$\hat{c} = c_0$$

- This is in contrast with the optimal $c^*$ obtained via the golden rule savings
- Assumptions on population growth, technical change and **resources** provide different results.

# Constant case

- Assume in the Solow-type model:
  - Population is constant, $L = const$;
  - Technology is constant, $A = const$;
  - No scarce resources.
- In this case

$$Y = F(AL, K) = ALf(k) = C + \dot{K}.$$

- Since $L = const$, consumption is constant, $C = const$;
- The largest (constant in time) consumption is:

$$C_{max} : \dot{K} = 0.$$

# Population growth

- Assume now $L = L_0 \mathrm{e}^{nt}$;
- One has standard Solow model:

$$\dot{k} = f(k) - nk - c;$$

- Golden rule capital and consumption are:

$$\hat{k} : f'(\hat{k}) = n,\ \hat{c} = f(\hat{k}) - n\hat{k};$$

- At the same time the maximin principle requires:

$$c^* = c_0 = f(k_0) - nk_0,$$

with $k_0 < \hat{k} \rightarrow c^* < \hat{c}$.

# Technical change

- Assume now ongoing labour-augmenting technical progress,

$$Y = F(K, \mathrm{e}^{at}L) = \mathrm{e}^{at}Lf(z);$$

  where $z$ is capital per labour in efficiency units, $z = K/\mathrm{e}^{at}L$.
- The growth equation is then

$$\dot{z} = f(z) - (n + a)z - c\mathrm{e}^{at}.$$

- Problem: find maximal $c = const$, such that $z(t) \geq 0 \forall t$.

# Problems

In the presence of ongoing technical progress constant consumption is **inefficient**:

- Choosing $c = c_0 = f(k_0) - nk_0$ means over-investments in the long-run (since technical change);
- Choosing $c = \hat{c} : f'(\hat{k}) = n + a$ leads to negative capital growth and economy collapses in finite time;
- However, there exists some $c : \lim_{t\to\infty} z(t) = 0$ which is maximal;
- This level of constant consumption leads to **consuming initial capital**.

# Solow and population

**"In a model with finite natural resources it seems ridiculous to hold to the convention of exponentially growing population. We all know that population can not grow forever, if only for square-footage reasons"**

# Population growth

- In this model population growth spoils everything;
- If $\dot{L} = n > 0$, then:
  1. Capital accumulation is bounded, since $y(t)$ goes to zero;
  2. This is true even with ongoing technical change;
  3. No positive constant consumption level may be found;
  4. The system is dynamically inconsistent, exactly in the spirit of Limits to Growth
- This result is independent of technical change;
- This does not include resource-replacing technical change though.

# Ongoing technical change

**"Unlimited technological progress may be unlikely, but it is not, like unlimited population growth on a finite planet, absurd."**

- If population is constant;
- If technical change is unbounded;
- Then:
  1. There exists maximal constant consumption level, $C_1 > C_0$;
  2. With this consumption both capital and resource stock are going to zero,

$$\lim_{t\to\infty} z(t) = 0, \ \lim_{t\to\infty} y(t) = 0, \tag{8}$$

  3. Minimax constant consumption is unsatisfactory.

# Discussion on Solow-Swan theory

- Simple model, unique steady state, still popular
- The crucial role of technical change and population growth
- Resources appear as a production factor
- They cannot be replaced by capital
- Pollution is not discussed (optimal control is necessary)

# Empirical evidence

- Some papers claim Solow model fits the data: Mankiw, Romer and Weil (1992)
- Others find out poor relevance with the data: Nakamura (2001)
- Both strands focus on the key role of **human capital** and **technology diffusion**
- Technology plays a key role, yet it is exogenous and static in Solow model!
- Central point is the **convergence issue**

# Reading


- Solow R. (1956). A Contribution to the Theory of Economic Growth. *The Quarterly Journal of Economics, 1(70)*, 65-94;
- Solow R. (1957). Technical Change and the Aggregate Production Function. *The Review of Economics and Statistics, 3(39)*, 312-320;
- Solow R. (1974) Intergenerational Equity and Exhaustible Resources. *The Review of Economic Studies, Vol. 41, Symposium of the Economics of Exhaustible Resources, pp. 29 - 45*;

# Next lecture

- Introduction to optimal control: Ramsey-Cass-Koopmans model (1965)
- Inclusion of technical change
- Optimal management of resources (based on R-C-K model): Dasgupta& Heal (1974)
- Combination of technical change and exhaustible resource in the model

# Eigenvalues computation

Back

- Given matrix $A$
- Compute **determinant** of $(A - \lambda I)$ where $I$ is identity matrix
- **Eigenvalues** of $A$ are roots of the polynomial

$$\det(A - \lambda I) = 0$$

- For 2$x$2 matrix $A$ it is a second degree:

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{23} - \lambda) - a_{12}a_{22} =$$
$$\lambda^2 - (a_{11} + a_{23})\lambda + a_{11}a_{23} - a_{12}a_{22} = 0$$

- With roots given by usual quadratic formula.

# Derivation of intensive form for capital accumulation

Back

$$\dot{K}(t) = L_0 \mathbf{e}^{nt} (\dot{K/L}) + n(K/L) L_0 \mathbf{e}^{nt};$$

$$\left(\dot{k}(t) + nk(t)\right) L_0 \mathbf{e}^{nt} = sF(K(t), L(t)) =$$

$$= sL_0 \mathbf{e}^{nt} F\left(\frac{K(t)}{L_0 \mathbf{e}^{nt}}, 1\right);$$

$$\dot{k}(t) = sF(k(t), 1) - (n + \delta)k(t). \tag{9}$$

# Golden rule derivation

Back

- Steady state definition:

$$\hat{k}: \ \dot{k} = sk^{\alpha} - (n+\delta)k = 0$$

- Which implies

$$\frac{\hat{k}}{\hat{k}^{\alpha}} = \frac{s}{n+\delta} \rightarrow \hat{k} = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

# Golden rule savings and stability

Back

- Golden rule capital implies

$$c = (1-s)\hat{k}^{\alpha} = (1-s)\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

- Maximization implies

$$\max c : \frac{d}{ds}c = 0 = \frac{\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}(s-\alpha)}{s(\alpha-1)} \rightarrow \hat{s} = \alpha$$

- Jacobian of (1) is

$$J = \{\alpha s k^{\alpha-1} - (n+\delta)\} \stackrel{k=\hat{k}}{=} (\alpha-1)(n+\delta)$$