Introduction to optimal control in growth theory

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Main features

- Multi-stage decision-making;
- Optimization of a dynamic process in time;
- Optimization is carried over **functions**, not **variables**;
- The planning horizon of an optimizing agent is taken into account (finite or infinite);
- The problem includes **objective** and the **dynamical system**;
- Some initial and/or terminal conditions are given.

Continuous-time problems

- Assume there is **continuous** number of stages (real time);
- **State** is described by continuous time function, $x(t)$;
- Initial and terminal states are fixed, $x(0) = x_0, x(T) = x_T$;
- Find a function $x(t)$, minimizing the cost of going from x_0 to x_T;
- What gives the costs?
- Concept of **objective functional**:

$$\min_u \int_0^T \left\{x(t) + u^2(t)\right\} dt$$

Ingredients of dynamic optimization problem

Every dynamic optimization problem should include:

- Some set of **boundary conditions**: fixed starting and/or terminal points;
- A set of **admissible paths** from initial point to the terminal one;
- A set of **costs**, associated with different paths;
- An **objective**: what to maximize or minimize.

Functionals

Definition

A **functional** J is a mapping from the set of paths $x(t)$ into real numbers (value of a functional).
$J := J(x(t))$.

- Functional is NOT a function of t;
- $x(t)$ is the *unknown* function, which have to be found;
- This is defined in some *functional space* $\mathcal{H}$;
- Hence formally $J : \mathcal{H} \to \mathbb{R}$.

Types of boundary conditions

1. **Fixed-time problem**: $x(0) = x_0$, time length is fixed to $t \in [0, .., T]$, terminal state is not fixed
   - Optimal price setting over fixed planning horizon
2. Fixed endpoint problem: $x(0) = x_0, x(T) = x_T$, but terminal time is not fixed
   - Production cost minimization without time constraints
3. Time-optimal problem: $x(0) = x_0, x(T) = x_T,\ T \rightarrow \min$
   - Producing a product as soon as possible regardless of the costs
4. Terminal surface problem: $x(0) = x_0$, and at terminal time $f(T) = x(T)$

In this course we mainly employ only type 1 with $T \rightarrow \infty$.

Transversality

- In variable endpoint problems as above given boundary conditions are not sufficient to find the optimal path
- Additional condition on trajectories is called **transversality** condition
- It defines, how the trajectory crosses the boundary line
- The vast majority of economic problems use this type of conditions
- Example: shadow costs of investments at the terminal time should be zero.

Problem

The subject of optimal control is:
Maximize (minimize) some objective functional

$$J = \int_0^T F(x(t), u, t)dt$$

with conditions on:

- Initial, terminal states and time;
  $x(0) = x_0; x(T) = x_T, t \in [0..T]$
- Dynamic constraints (define the dynamics of states);
  $\dot{x}(t) = f(x, u, t)$
- Static constraints on states (nonnegativity, etc.)
  $x(t) \geq 0, u(t) \geq 0.$

Hamiltonian

- To solve an optimal control problem the Hamiltonian function is needed
- This is an equivalent of Lagrangian for static problems
- It includes the objective and dynamic constraints
- If static constraints are present, the augmented Hamiltonian is used
- First order conditions on Hamiltonian provide optimality criteria.

Construction

Let the optimal control problem be:

$$J := \int_0^T F(x,u,t)dt \to \max_u;$$
$$s.t.$$
$$\dot{x} = f(x,u,t). \quad (1)$$

Then the associated Hamiltonian is given by:

$$\mathcal{H}(\lambda, x, u, t) = F(x,u,t) + \lambda(t) \cdot f(x,u,t). \quad (2)$$

Comments

- In the Hamiltonian $\lambda(t)$ is called **costate** variable;
- It usually represents shadow costs of investments;
- Investments are **controlled**, $u(t)$;
- This has to be only piecewise-continuous and not continuous;
- Number of costate variables = Number of dynamic constraints the system has;
- Unlike lagrange multipliers, costate variable changes in time;
- The optimal dynamics is defined by the pair of ODEs then: for **state**, $x(t)$ and **costate**, $\lambda(t)$.

Example

Consider the problem:

$$\max_{u(\bullet)} \int_0^T e^{-rt}[-x(t) - \frac{\alpha}{2}u(t)^2]dt$$

$$\begin{aligned} s.t. & \\ & \dot{x}(t) = \beta(t) - u(t)\sqrt{x(t)}, \\ & u(t) \geq 0, x(0) = x_0. \end{aligned} \tag{3}$$

where $\beta(t)$ is arbitrary positive-valued function and α, r, T are constants.

The Hamiltonian of the problem (3) should be:

$$\mathcal{H}^{CV}(\lambda, x, u, t) = -x - \frac{\alpha}{2}u^2 + \lambda^{CV}[\beta(t) - u\sqrt{x}]. \qquad (4)$$

Where the **admissible set** of controls include all nonnegative values $(u(t) \geq 0)$.
QUESTION: where is the discount term e^{-rt}?
Transformation
$e^{-rt}\lambda(t) = \lambda^{CV}(t)$ yields
current value Hamiltonian.
It is used throughout all the economic problems.

Optimality conditions

The **optimal** control $u(t)$ is such that it maximizes the Hamiltonian, (2), and

$$u^* : \frac{\partial \mathcal{H}(\lambda, x, u, t)}{\partial u} = 0;$$
$$\mathcal{H}(\lambda, x, u, t) = \mathcal{H}^*(\lambda, x, t) \tag{5}$$

must hold for *almost all* t.
This is **maximum condition**.
Along optimal trajectory

$$\dot{\lambda}(t) = r\lambda(t) - \mathcal{H}_x^*(\lambda, x, t). \tag{6}$$

which is the **adjoint** or **costate** equation, and

$$\lambda(T) = 0 \tag{7}$$

which is transversality condition.

Sufficiency

- The conditions above provide only necessary, but not sufficient criteria of optimality
- The sufficient condition is given by the **concavity** of a maximized Hamiltonian $\mathcal{H}^*$ w. r. t. $x(t)$
- Once the Hamiltonian is linear in state and quadratic in control, it is always concave
- Sufficient condition is thud satisfied
- This is always true for **linear-quadratic** problems.

Main points on optimal control

To **solve** an optimal control problem is:

- Right down the Hamiltonian of the problem;
- Derive first-order condition on the control;
- Derive costate equation;
- Substitute optimal control candidate into state and costate equations;
- Solve the canonical system of equations;
- Define optimal control candidate as a function of time;
- Determine the concavity of a maximized Hamiltonian (usually neglected).

Roots

- The initial Ramsey model (1928) was the first optimization-type macroeconomic model
- He asks the question "How much of its income should the nation save?"
- The **dynamic** choice between consumption and savings in order to maximize utility
- Only one good, and only one representative agent
- Infinite time-horizon and no discount rate at all
- There is a static choice between consumption and labour, but no explicit production function
- Utility is separable in consumption and labour
- This was adapted by Cass and Koopmans for neoclassical growth theory in 1965.

Assumptions

- Large number of identical firms
- Two production factors: L, K
- Constant returns to scale production technology
- Firms maximize profits and are owned by households
- Identical households
- They supply labour (one unit per household) and rent capital to firms
- Household divides its income between consumption and capital investments
- Objective is to maximize life-time utility of the (representative) household **choosing dynamic consumption profile**.

Formulating the Dynamic Problem

Production function:

$$Y = F(K, L)$$

It is then rewritten in intensive form with usual properties:

$$f'(k) > 0, f''(k) < 0, \lim_{k \to 0} f'(k) = \infty, \lim_{k \to \infty} f'(k) = 0.$$

Net investments can be expressed as:

$$I = \dot{K}(t) = Y(t) - C(t) - \delta K(t).$$

In per capita terms this yields **dynamic constraint**

$$\dot{k} = f(k) - c - (n + \delta)k.$$

similar to Solow model

Problem

The model is formulated as **optimal control problem** of the social planner:

$$J := \int_0^\infty \mathbf{e}^{-rt} U(c) dt \to \max_c$$

$$s.t.$$

$$\dot{k} = f(k) - c - (n+\delta)k;$$

$$k(0) = k_0;$$
$$0 \leq c \leq f(k). \quad (8)$$

This is an optimal control problem with one **state** variable and one **control** variable.

Hamiltonian construction

Hamiltonian for the Problem (8) is straightforward:

$$\mathcal{H} = U(c)\mathbf{e}^{-rt} + \lambda[f(k) - c - (n+\delta)k]$$

or, alternatively, current-value Hamiltonian:

$$\mathcal{H}^{CV} = U(c) + \lambda^{CV}[f(k) - c - (n+\delta)k] \tag{9}$$

this do not include the control constraint. With its inclusion one has augmented Hamiltonian:

$$\mathcal{H}_A^{CV} = U(c) + \lambda^{CV}[f(k) - c - (n+\delta)k] + \mu[f(k) - c]. \tag{10}$$

Obtaining dynamics

Using Pontryagin's Maximum Principle, we have:
Maximum condition as in (5):

$$\frac{\partial \mathcal{H}^{CV}}{\partial c} = U'(c) - \lambda(t)^{CV} = 0; \tag{11}$$

Costate equation as in (6):

$$\dot{\lambda}(t)^{CV} = r\lambda^{CV}(t) - \frac{\partial \mathcal{H}^{CV}}{\partial k} = -\lambda^{CV}(t)[f'(k(t)) - (n+\delta+r)]; \tag{12}$$

And state equation

$$\dot{k}(t) = f(k(t)) - c - (n+\delta)k(t). \tag{13}$$

Canonical system

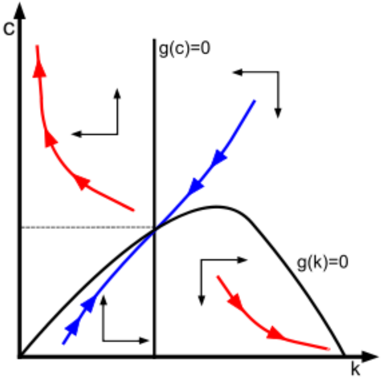
- Since $U(c)$ is general form, optimal control cannot be defined;
- Rather we eliminate costate from the system;
- One obtains the dynamics as a pair of equations in c and k:

$$\dot{k} = f(k) - c - (n+\delta)k;$$
$$\dot{c} = -\frac{U'(c)}{U''(c)} \cdot [f'(k) - (n+\delta+r)];$$
$$-\frac{U'(c)}{U''(c)} > 0. \tag{14}$$

Qualitative analysis

Steady states are defined by zero growth of both variables:

$$\begin{aligned} c &= f(k) - (n+\delta)k \Leftrightarrow g(k) = 0; \\ f'(k) &= n + \delta + r \Leftrightarrow g(c) = 0. \end{aligned} \tag{15}$$



Steady states comparison

- Quadrants are defined by steady state conditions on c and k;
- Their intersection provides the unique fixed point of the system;
- The capital level associated with this fixed point is known as the modified golden rule level.

$$\bar{k} : f'(\bar{k}) = n + \delta + r < \hat{k} : f'(\hat{k}) = n + \delta. \tag{16}$$

- Consumption level is thus also lower than for the basic Solow model

Phase space analysis

To define the dynamics of the system $c - k$ in different regions of the phase space, evaluate the derivatives:

$$\frac{\partial \dot{k}}{\partial c} = -1 < 0;$$

$$\frac{\partial \dot{c}}{\partial k} = -\frac{U'(c)}{U''(c)} f''(k) < 0. \tag{17}$$

The more formal way (and valid for any dimension!) of analysing stability and dynamics is through the Jacobian matrix of the system.

Additional notation

Now we include technical progress into the basic Ramsey-Cass-Koopmans model.
This is done in the same way as for the Solow model:

$$\eta = AL;$$
$$Y = Y(K, \eta).$$

We define the efficient labour as an input rather than "true" labour. Then proceed in the same way as before:

$$y_\eta = f(k_\eta),$$
$$y_\eta = \frac{Y}{\eta}, k_\eta = \frac{K}{\eta}, c_\eta = \frac{C}{\eta}, a = \frac{\dot{A}}{A}.$$

We have the same capital-intensive variables just as in Solow model with technical change.

Modified Problem

Modifying the equation of motion we have almost the same problem, as Problem (8), but with modified capital per effective labour unit variable:

$$J_\eta := \int_0^\infty \mathbf{e}^{-rt} U(c_\eta) dt \to \max_{c_\eta}$$

s.t.

$$\dot{k}_\eta = f(k_\eta) - c_\eta - (a + n + \delta) k_\eta;$$

$$k_\eta(0) = k_{\eta,0};$$
$$0 \leq c_\eta \leq f(k_\eta). \tag{18}$$

Hamiltonian

The (current-value) Hamiltonian of the Problem (18) is also almost the same:

$$\mathcal{H}_{\eta}^{CV} = U(c_{\eta}) + \lambda_{\eta}^{CV}[f(k_{\eta}) - c_{\eta} - (a + n + \delta)k_{\eta}] \tag{19}$$

With maximum condition:

$$U'(c_{\eta}) = \lambda_{\eta}^{CV}. \tag{20}$$

Dynamical System

With the same procedure of replacing costate with consumption share, we have the 2-dimensional system for modified capital and consumption shares:

$$\dot{k}_\eta = f(k_\eta) - c_\eta - (a + n + \delta)k_\eta;$$

$$\dot{c}_\eta = -\frac{U'(c_\eta)}{U''(c_\eta)}[f'(k_\eta) - (a + n + \delta + r)]. \tag{21}$$

which differs in the additional technology term a.

Differences in dynamics

- One has the same phase diagram as for the basic model;
- Steady state levels of k, c are also defined similarly;
- However, the steady-state values are different because of technical change:

$$\bar{c}_\eta = const = \frac{C}{AL} \tag{22}$$

  and hence the consumption share per real physical worker is NOT constant:

$$\bar{c} = \frac{C}{L} = \bar{c}_\eta \times A \neq const. \tag{23}$$

- we have now **ongoing growth** with rising consumption per worker.

Discussion

- Neoclassical models of growth do not allow *per se* for ongoing growth in intensive terms;
- Such a rise in per capita consumption has been introduced through **technical change**;
- Since then terms of growth and technical change are interrelated;
- Technical change is exogenous, unexplained;
- It affects only labour productivity;
- Consumption grows at exactly the same rate as the technical change (labour productivity);
- Only one control parameter: per capita consumption.

Reading

- Ramsey F. (1928) A Mathematical Theory of Saving. *The Economic Journal, 38 (152)*: 543-559;
- Cass, David (1965). Optimum Growth in an Aggregative Model of Capital Accumulation. *Review of Economic Studies 32 (3)*: 233240;
- Koopmans, T. C. (1965). On the Concept of Optimal Economic Growth. *The Economic Approach to Development Planning. Chicago: Rand McNally*: 225287;

Next lecture

- Competing views: Market is sufficient (Coase) vs. Market is insufficient
- Paper: Keeler, Spence, Zeckhauser (1972)
- How we can include pollution in the neoclassical framework?
- What is the optimal management of pollution?
- Is it different from resource management?
- The role of social planner