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Vincenz Bronzin's Option Pricing Theory: Contents, Contribution, and Background

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Vincenz Bronzin's Option Pricing Theory:
Contents, Contribution, and Background

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[#] This paper originates in an email sent by the second author wondering whether the first author knew about Bronzin's booklet on option pricing, dating back almost a century and containing formulas which appear rather similar to those developed by Black-Scholes. The scepticism of the first author quickly disappeared after reading Bronzin's manuscript. - We would like to thank Robert C. Merton for clarifying comments. The detailed and insightful suggestions by David Rey, Yvan Lengwiler and Stefan Duffner have substantially improved the paper. Ana Perisic established the contact to Gerhard Bronzin. In Trieste, the following persons were extremely helpful, with respect to contacts, information, and comments: Giorgio Raldi, Anna Millo, Anna Maria Vinci, Ermanno Pitacco, Patrik Karlsen, Clara Gasparini (RAS) and from Generali: Alfred Leu, Alfeo Zanette, Marco Sarta, Ornella Bonetta (Biblioteca). The staff of the Archivio di Stato di Trieste, of the Biblioteca Civica di Trieste, and the Biblioteca dell'Assicurazioni Generali, Trieste, was extremely helpful and supporting. Partial financial funding by the WWZ-Förderverein is gratefully acknowledged under the project no. B-086.

1. Introduction

The doctoral thesis of Louis Bachelier (1900) is widely considered as the seminal work in option pricing theory. However, only a few years later, 1908, Vinzenz Bronzin, who was a professor of mathematics at the Accademia di Commercio e Nautica in Trieste, published a booklet in German, some 80 pages long, entitled *Theorie der Prämiengeschäfte* (Theory of Premium Contracts). While the work got some attention in the academic literature soon after it was published, it seems to have almost been forgotten later[1], and more recent academic mentions are virtually inexistent[2].

While his approach is more pragmatic than Bachelier's, every element of modern option pricing can be found: Risk neutral pricing, no-arbitrage and perfect-hedging pricing conditions, the put-call-parity, and the impact of different distributional assumptions on option values. In particular, he shows how the normal law of error – which is the normal density function – can be used to price options, and how it is related to a binomial stock price distribution. His equation (43) is closer to the Black-Scholes formula than anything published before Black, Scholes, and Merton. He moreover develops a simplified procedure to find analytical solutions for option prices by exploiting a key relationship between their derivatives (with respect to their exercise prices) and the underlying pricing density. Besides of pricing simple calls and puts, he develops formula for chooser options and, more important, repeat-options.

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[1] For example it was mentioned in a standard banking textbook from Friedrich Leitner (1920) and a book by Karl Meithner (1924) The most notable exception is a follow-up paper by the mathematician Gustav Flusser (1911) extending some of Bronzin's results. Moreover, the book got a short review in the famous *Monatshefte für Mathematik und Physik* in 1910 (Volume 21).

[2] Except a recent reference from our colleague Yvan Lengwiler (2004), we are aware of only one modern reference on Bronzin's book in a German textbook on option pricing (see Welcker et al. 1988). The authors do not comment on the significance of Bronzin's contribution in the light of modern option pricing theory. A short appreciation of Bronzin's book is also contained in a recent monograph of one of the authors of this paper, Hafner (2002).

Our "discovery" also raises questions beyond the analytics: why did the results of Bachelier, Bronzin, and possibly other's yet to be re-discovered, not get a broader acceptance? Why did their research not find immediate successors, academics that made it a subject of ongoing scientific research? Finding answers to these questions could help us to better understand the cultural background of financial mathematics, and would probably add an interesting chapter to the sociology of science.

A general difficulty in the attempt to write about Bronzin's book is that the text is written in German, and many of his finance related expressions (which may or may not reflect the commonly used terms at the time being) cannot be translated easily. We therefore have to find English terms as adequate as possible, and add the original German wording in parentheses where it seems to be useful[3]. Moreover we have adapted Bronzin's mathematical notation with only minor changes. In discussing, or extending certain results (particularly in section 5, subsection 5.6), we have tried to make a clear distinction between the results of Bronzin and our own.

The structure of the paper is as follows: Section 2 describes the basic terminology as well as the range of derivative contracts analysed in Bronzin's book. Section 3 is about his thoughts on hedging, replication, and arbitrage – although he does not formally use these terms. Section 4 outlines the major elements of his valuation approach: the probabilistic foundations, zero-profit conditions and a characterization of the (risk neutral) pricing density. Section 5 gives an overview on the major section of the book, namely the derivation of option prices under alternative specifications of the probability (or pricing) density function. This section also shows the close relationship between one of these specifications, the error function, and the Black-Scholes / Merton model. Section 6 tries to make an overall as-

[3] Occasionally, interested readers find important sentences in the full original German wording in footnotes.

sessment of the scientific contribution of Bronzin's book in the light of the history of option pricing. Finally, in Section 7, we give a brief description of the scientific and socio-cultural background of Bronzin's work and professional activities, which also includes thoughts about the state of probabilistic thinking in physics and actuarial science as the key analytical prerequisites of modern option pricing.

It should be mentioned that this paper is only a partial appreciation of Bronzin's work on option pricing. We try to highlight the most important elements of his analysis. A more complete characterization is provided by Hafner/ Zimmermann (2004). Also, the content of Section 7 is rather preliminary – and reflects ongoing research which is far from being completed. Nevertheless, we think that it is important to highlight the cultural, social and scientific role of the k.u.k.-Academy in Trieste within the Hungarian-Austrian empire, where V. Bronzin started his career at young age and was affiliated until he passed away in 1970, at 98 years of age.

2. *Basic structure and terminology*

2.1 *Structure of the book*

Bronzin's book contains two major parts. The *first* part is more descriptive and contains a characterization and classification of basic derivative contracts, their profit and loss diagrams, and basic hedging conditions and (arbitrage) relationships. The *second*, and more interesting part is on option pricing and starts with a general valuation framework, which is then applied to a variety of distributions for the price of the underlying security in order to get closed form solutions for calls and puts. Among these distributions is the "error function" which is closely related to the normal distribution. It is interesting to notice that the separation of topics between "distribution-free" and "distribution-related" results is in perfect line with the modern classification of option pricing topics, following Merton (1973).

In this part, Bronzin's methodological setup is completely different from Bachelier's, at least in terms of the underlying stochastic framework. He develops no stochastic process for the underlying asset price and uses no stochastic calculus, but directly makes different assumptions on the share price distribution at maturity and derives a rich set of closed form solutions for the value of options. This simplified procedure is justified insofar as his work is entirely focused on European style contracts (not to be exercised before maturity), so intertemporal issues (e.g. optimal early exercise) are not of premier importance.

2.2 *Contracts and basic terminology*

The analysis of Bronzin covers forward contracts as well as options, but his main focus is on the latter. The term "option" does not show up. Instead, his analysis is on "premium contracts" (*Prämiengeschäfte*) which is an old type of option contract used in many European countries up to the seventies, before warrants and traded options became popular[4]. The buyer of a premium contract has the right to step down from a forward contract *at* maturity – not before, so the contract is always European style. The four resulting positions are clearly characterized, analytically as well as in terms of payoff diagrams (pp. 2-7). The buyer of a premium contract acquires either the right to buy (*Wahlkauf*) or to sell (*Wahlverkauf*) the underlying security at maturity, while the seller of the contract has the obligation to sell (*Zwangsverkauf*) or buy (*Zwangskauf*) the underlying. Forward contracts are

[4] There is a paper on the French premium market by Courtadon (1982).

called "fixed contracts" (*Festgeschäfte*) in Bronzin's terminology (pp. 1-2).

Within the category of option contracts, Bronzin distinguishes between "normal" and "skewed" (*schiefe*) contracts. A normal option contract exhibits an exercise price equal to the forward price, the latter being denoted by B throughout the book. Because of obvious reasons, we will refer to this case as "at-the-money" (ATM) contracts. Skewed call and put contracts exhibit exercise prices which deviate by a magnitude M from the forward price. We will denote exercise prices by K in this paper, which implies $K \equiv B + M$.

In addition to these standard (or simple) options, Bronzin analyses two special contracts: choser options (called *Stella-Geschäfte*) where the buyer has the right to determine whether he wants to buy or sell the underlying at maturity[5]; and a special kind of "repeat option" (called *Noch*[6]*-Geschäft*) which adds a (multiple) option component to a forward transaction. The latter contract will be analyzed briefly in Section 6[7].

Throughout the book, Bronzin does not refer to a specific underlying security in his analysis[8], nor to other institutional characteristics of the contracts he analyses. The underlying security is often just called "object" (*Wertobjekt*), and its price is referred to as "market" price.

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[5] They are also shortly addressed by Bachelier; see (p. 53) on *double primes*.
[6] The German word "*noch*" is uncommonly used as a noun here; in common language it is a pronoun and means "another", i.e. an additional one of the same kind, one more.
[7] Also these contracts are shortly analysed by Bachelier; see Section 6 for a direct comparison.
[8] Except in the final numerical example on the second-last page, where he refers to "shares" (*Aktien*).

3. *Valuation fundamentals: Hedging, replication, and arbitrage*

Two key concepts, "coverage" (*Deckung*) and equivalence (*Äquivalenz*), play an important role in the first part of Bronzin's book (see sections 4 and 5 in chapter I, and section 3 in chapter II). Although the focus of the author is not always clear, this part of the text is nevertheless interesting because, in the light of modern[9] option pricing theory, it is an early notion of perfect hedging and replication of option positions, and the conditions for their general feasibility. Unfortunately, at this stage of analysis, the author does not introduce the concept of arbitrage (or what he later calls "fair pricing"), but discusses the pricing implications rather as "full hedging conditions".

Bronzin defines a "covered" position as a combination of transactions (options *and* forward contracts) which is immune against profits and losses[10]. Two systems of positions are called "equivalent" if one can be derived (*abgeleitet*) from the other, or stated differently, if they provide exactly the same profit and loss for all possible states of the market[11]. From a linguistic point of view, it is interesting to notice that Bronzin explicitly uses the word "derived" in this context.

Bronzin also stresses the relationship between the two concepts: One can always get two systems of equivalent transactions if we take a subset of contracts within a complex of covered transactions and reverse their signs[12]. This basic insight is then followed by a lengthy characterization of conditions under which combined call and put

[9] In this paper, „modern" option pricing always refers to the state of the theory after the Black-Scholes-Merton breakthrough.

[10] Original text: „*Wir werden einen Komplex von Geschäften dann als gedeckt betrachten, wenn bei jeder nur denkbaren Marktlage weder Gewinn zu erwarten noch Verlust zu befürchten ist*" (p. 8).

[11] Original text: "*Zwei Systeme von Geschäften nennen wir nämlich dann einander äquivalent, wenn sich das eine aus dem anderen ableiten lässt, in anderen Worten, wenn dieselben bei jeder nur dankbaren Lage des Marktes einen ganz gleichen Gewinn resp. Verlust ergeben*" (p. 10).

[12] Original text: "*... dass wir sofort zwei Systeme äquivalenter Geschäfte erhalten, wenn wir nur in einem Komplexe gedeckter Geschäfte einige derselben mit entgegengesetzten Vorzeichen betrachten*" (p. 10).

option positions can be fully "covered" (hedged) – by large systems of equations, which are not easily accessible.

In this context, the put-call-parity[13] is derived:

i) For symmetric i.e. ATM call and put positions (chapter 1, Section 4): The number[14] of long and short options must be equal, *and* the (net or residual[15]) number of long call (put) options must be matched by the same number of forward sales (buys)[16]. Moreover, the call and put price must be equal.

This is a slightly complicated way to state that a long position of calls (puts) plus a short position of puts (calls) produces a synthetic long (short) forward contract. The more interesting part in the statement is the equivalence of option prices

(1)… $$C[K = B] = P[K = B]$$

(K denotes the exercise price) which is a special case of the well-known put-call parity

ii) For skewed positions, i.e. calls and puts with arbitrary but equal exercise price (chapter 2, Section 1): The same conditions as before must hold, but the equality of call and put price is replaced by the "remarkable" (*bemerkenswerte*) condition

[13] Although not explicit in Bronzin's text, the subsequent hedging conditions refer to options on the same underlying with the same maturity.
[14] Bronzin argues in terms of the „number" of options, but obviously, he assumes equal dollar amounts and equal exposures (which is trivially the case since all options have the same exercise price, B, by assumption).
[15] This specification is not done by Bronzin, but is obvious.
[16] Original text: „*Es müssen ... Wahlgeschäfte in gleicher Anzahl wie Zwangsgeschäfte vorkommen; zu gleicher Zeit müssen aber ... ebenso viele feste Verkäufe desselben Objekts vorgenommen werden, als Wahlkäufe vorhanden sind, oder, was auf dasselbe hinauslaufen muss, ..., ebensoviel feste Käufe abgeschlossen werden, als Wahlkäufe vorhanden sind*" (p. 9).

(2)… $P[K = B + M] = C[K = B + M] + M$

which is the put-call-parity, because M measures the "moneyness" of the options. For $M > 0$, the put option is in-the-money, and the equation shows that the put price exceeds the call price by exactly the amount of the moneyness, M. The reverse is true if $M < 0$.

It is important to notice that Bronzin derives this parity relationship as a necessary condition for the feasibility of a perfect hedge[17]. It is apparently obvious for him that a position which is fully hedged against all states of the market cannot exhibit a positive price – but there is no explicit statement of this kind.

Relating his equation to the standard formulation of the put-call parity, it is easy to recognize that the moneyness of the (put) option, M, must just be specified in terms of current dollars as

$$M = Ke^{-r(T-t)} - S_t$$

to get the traditional parity relationship (r is the continuously compounded riskfree rate, $T-t$ time to maturity, and S_t the current stock price.

Unfortunately, the notion of arbitrage does not show up explicitly in Bronzin's text[18] - but he is close to it. In part I, the put-call-parity is derived as part of the perfect hedging condition for joint put/call positions, without assuming a specific probability distribution for the future market price. It is a distribution-free result. How-

[17] See e.g. his remark: "*Es müssen überdies zwischen den Prämien der Wahlkäufe und Wahlverkäufe, damit überhaupt eine Deckung möglich ist, die aufgestellten Bedingungen … eingehalten werden* ..." (p. 18).
[18] The same is also true for Bachelier; amazingly, Bronzin published a paper entitled "arbitrage" a few years earlier; Bronzin (1904).

ever, no explicit mention is made about arbitrage in the modern sense of the word[19].

In part II (chapter 1, p. 44 for symmetric contracts, p. 47 for asymmetric contracts) the put-call-parity is again derived, but this time from an explicit *pricing* relationship: Based on a particular price distribution, Bronzin postulates a general valuation principle according to which no profit or loss should be expected for any of the two parties (the buyer and the seller) involved in the transaction when the contract is negotiated. This is a "fair pricing" or "zero expected profit" condition, but no non-arbitrage condition because it refers to expectations, not immediate (risk-free) profits as required by arbitrage. However, Bronzin recognizes that this derivation has a different qualitative nature than in the previous part: the parity relationship has now no longer the character of an artificial condition but emerges from the "incontestable" principle of reciprocity in business transactions[20]. Of course, his remark that the parity only gets its "full justification and importance" at this stage of analysis is not correct, because the derivation in a distribution-free setting is more general. But Bronzin apparently recognizes that deriving the parity by using some kind of "equilibrium" relation adds a new dimension to the pricing of options – although this is not necessary for the put-call-parity itself, but for the other pricing relationships he is about to derive.

[19] An arbitrage profit is riskfree and does not require a positive amount of capital being invested.
[20] This is a rather free translation. The orginal text reads as: "... *sondern dem unanfechtbaren Prinzip der Gleichheit von Leistung und Gegenleistung entsprungen ist*" (p. 48).

4. The probabilistic setting and general valuation framework

Bronzin recognizes that his analysis in part I of his booklet leaves open the fundamental question about the appropriate (*rechtmässig*) size of the option premiums. He also recognizes that further assumptions and tools[21] are necessary to achieve this goal: probabilistic assumptions about the market[22], and a rule to translate expected profits and losses from the contracts to current values.

4.1 The probability density

The market model which Bronzin has in mind can be characterized as a driftless random walk. In this respect, the approach is virtually identical to Bachelier (1900).

- *Random walk*. When discussing the possible specification of the probability density function of the underlying market price (p. 56), he finds himself in substantial difficulties: He argues that he does not know any general criteria to characterize the random (*regellos*) market movements for the various underlyings analytically[23]. Instead, he proposes to statistically estimate possible distributions (see Section 5).

- *Spot and forward price*. The starting point of Bronzin's probabilistic market model is the forward price B. He assumes that this price is "naturally" close or even identical to the current spot

[21] He notices that the tools which are required for this task are beyond elementary mathematics – only the application of probability and integral calculus is able to shed light on this important question (p. 39).
[22] It is interesting to notice that his focus is from the beginning on the variability (volatility) and the current state of the market (*Marktschwankungen*), not the trend.
[23] Original text: "*Allgemeine Anhaltspunkte, um die regellosen Schwankungen der Marktlage bei den verschiedenen Wertobjekten rechnerisch verfolgen zu können, gehen uns vollständig ab*" (p. 56).

price[24]. Since there is no mention about interest rates, the time value of money, or discounting anywhere in the book, this also implies that he assumes an efficient market.

- *Price expectation*. He repeatedly argues that the forward price is the most likely among all possible future market prices (p. 56, p. 74, p. 80), i.e. the forward price is an unbiased predictor of the future spot price. Otherwise, he argues, one could not imagine sales and purchases (i.e. opposite transactions) with equal chances if strong reasons would exist leading people to ultimately predict either a rising or falling market price with higher probability[25]. Thus, the forward price is regarded as the most advantageous price for both parties in a forward transaction[26]. A slightly different reasoning is used when discussing the payoff diagram of a forward contract, where he states that the forward price B must be such that the two "triangle parts" to the left and the right of B, i.e. to the profit and loss of the contract, must be "equivalent" because otherwise, selling or buying on spot should be more profitable[27]. This does not necessarily imply an unbiased forward price, although there is little doubt that he wants to claim this.

While the issue of price expectations seems to be important for Bronzin, it is not relevant for the development of his model. The important point is that the mean of the price distribution is based on *ob-*

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[24] Original text: „... *zum Kurse B, welcher natürlicherweise mit dem Tageskurse nahe oder vollkommen übereinstimmen wird....*" (p. 1).

[25] Original text: "*Es könnten ja sonst nicht Käufe und Verkäufe, d.h. entgegengesetzte Geschäfte, mit gleichen Chancen abgeschlossen gedacht werden, wenn triftige Gründe da wären, die mit aller Entschiedenheit entweder das Steigen oder das Fallen des Kurses mit grösserer Wahrscheinlichkeit voraussehen liessen*" (p. 74).

[26] On (p. 56), the reasoning for this insight is justified by the fact, that the call and put prices coincide if the exercise price is equal to the forward price.

[27] Original text: "*Es braucht kaum der Erwähnung, dass die dreieckigen Diagrammteile rechts und links von B als äquivalent anzunehmen sind, da sonst entweder der Kauf oder der Verkauf von Haus aus vorteilhafter sein sollte*" (p.1). The wording "von Haus aus" is no longer known in the German language, but it obviously means a spot transaction.

servable market price (spot or forward price), not price expectation or other preference-based measures[28].

However, whether the forward price matches the expected future price or not is not relevant for Bronzin's subsequent analysis. This would be relevant if statements about risk premiums or risk preferences should be made, which is not the intention of the author. Instead, his focus is on *consistent* (or in his wording, "fair") pricing relationships between spot, forward, and option contracts – which qualifies his probability density as a risk neutral density.

4.2 *Fair pricing: Zero expected excess returns*

As noted before, Bronzin understands the forward price as the cutting edge for modeling the ups and downs of the underlying market price. He consequently characterizes the random behavior of the market price by its deviation from the forward price, $\tilde{x} \equiv \tilde{S}_T - B$, where $\tilde{S}_T$ is the stock price at maturity (which is however never focused throughout the text). He also applies this characterization to his definition of expected profits and losses:

- *The "expected value" of a contract is zero.* Bronzin (pp. 41/42) states the important valuation principle that at contract settlement, no profit or loss should be expected[29] for any of the two parties (the buyer and the seller) involved in the transaction. For this purpose, the conditions of each transaction must be determined in a way that the sum of expected profits of both parties (taking

[28] The same is true for Bachelier's analysis. In contrast to Bronzin, he does not argue with the forward price, but he apparently assumes that the price at which a forward contract (*opération ferme*) is executed is equal to the current spot price (see his characterization on p. 26; notice that his x is the deviation of the stock price at expiration from the current value).

[29] It is important to notice that the statement, in the literal sense, is about expected, not current (riskless), profits. It is therefore not a no-arbitrage condition. Original text: "*... dass im Moment des Abschlusses eines jeden Geschäfts beide Kontrahenten mit ganz gleichen Chancen dastehen, so dass für keinen derselben* IM VORAUS *weder Gewinn noch Verlust anzunehmen ist*" (p. 42).

losses as negative profits) is zero[30]. Bronzin calls this the "fair pricing" condition (*Bedingung der Rechtmässigkeit*).

Notice that profits and losses are defined with respect to the forward price – which is a major difference to Bachelier's Martingale assumption which is defined relative to the current stock price[31]. Hence, Bronzin considers a pricing rule as "fair" if expected profits and losses of a contract are derived from a "pricing" density of the underlying which is centered at the forward price.

4.3 Substituting probabilities by prices

The most amazing part of Bronzin's booklet is in Section 8 of the first chapter in part II, where he relates the probability function $f(x)$ to option prices. This was explicitly done in an unpublished paper by Black (1974) [32], and a few years later by Breeden/ Litzenberger (1978). By referring to the rules of differentiation with respect to boundaries of integrals, and expressions within the integral (generally known as Leibnitz rules), he derives the "remarkable" expression

(5)… $$\frac{\partial P_1}{\partial M} = -\int_M^{\omega} f(x)dx = -F(M) \quad \text{(equation 16, p. 50).}$$

Remember that $F(M)$ is the probability that the stock price exceeds the exercise price at maturity, i.e. that the options gets exercised. Equation (5) thus postulates that the negative of the exercise probabil-

[30] Original text: "*wir stellen uns also jedes Geschäft unter solchen Bedingungen abgeschlossen vor, ... dass der gesamte Hoffnungswert des Gewinns für beide Kontrahenten der Null gleichkommen müsse*" (p. 42).
[31] For example: "*L'espérance mathématique du spéculateur est nulle*" (p. 18); "*Il semble que le marché, c'est-à-dire l'ensemble des spéculateurs, ne doit croire à un instant donné ni à la hausse, ni à la baisse, puisque, pour chaque cours coté, il y a autant d'acheteurs que de vendeurs*" (pp. 31-32); "*L'espérance mathématique de l'acheteur de prime est nulle*" (p. 33).

[32] Many years ago, William Margrabe made one of the authors, hein Zimmermann, aware of this paper. Not many people seem to know this tiny piece; e.g. it is also missing in the Merton and Scholes *Journal of Finance* tribute after Fischer Black's death, where all his papers and publications are listed.

ity is equal to the first derivative of the option price with respect to the exercise price (respectively, M). Remarkably, he notes that by this expression is much easier to solve for P_1 than in the standard valuation approach: Based on (5), the option price can be computed by the indefinite integral

$$(6)\ldots \qquad P_1 = -\int F(M)\,dM + c \quad \text{(equation 19, p. 51)}$$

where c is a constant which is not difficult to compute (it will be zero or negligible in most cases). Equation (6) is a powerful result: Option prices can be computed by integrating $F(M)$ over M. Depending on the functional form of $f(x)$, this could drastically simplify getting option values. Thus, knowing (or determining) the function $F(x = M)$ showing the exercise probabilities as a function of M, is the key element in determining option values under this approach. From there, it is straightforward to show that the second derivative

$$(7)\ldots \qquad \frac{\partial^2 P_1}{\partial M^2} = f(M) \quad \text{(equation 17, p. 51)}$$

directly gives the value of the (probability density) function at $x = M$[33]. As Breeden/ Litzenberger (1978) have shown, this derivative multiplied by the increment dM can be interpreted as the implicit state price[34] in the limit of a continuous state space. Bronzin also shows that equation (7) can be applied without adjustments to put options.

It is apparent from Bronzin's equations (16), (17) and (19), that he was aware that information on the unknown function $f(x)$ is impounded in observed (or theoretical) option prices, and just need to

[33] Interestingly, Bachelier (1900) on p. 51 also shows this expression, but without motivation, comments, or potential use.
[34] For a discrete distribution of states, the state price (also called Arrow-Debreu price) is the current price of a claim, which entitles its owner to receive a dollar in one specific future state, and nothing otherwise.

be extracted. It establishes $f(x)$ as a *pricing* function (or density), or to put it more directly: they demonstrate the key relationships between security prices and probability densities.

The analytical implications of equations (5) and (6) are of key interest to Bronzin, and we therefore provide a brief illustration using the "triangle distribution" which he uses later in his analysis. $f(x)$ is specified as a linear function $f(x)=a+bx$, defined over the interval $[0;+\omega]$; and respectively $f_1(x)=a+b|x|$ if x is in the negative range $[-\omega;0]$. For $f(\omega)=f_1(\omega)=0$ to hold, the parameters must be specified as $a=\frac{1}{\omega}$, $b=-\frac{1}{\omega^2}$, which implies $f(x)=\frac{\omega-x}{\omega^2}$.

The standard pricing approach requires the solution of the integral

$$P_1=\int_M^\omega (x-M)f(x)dx=\int_M^\omega (x-M)\frac{\omega-x}{\omega^2}dx$$

which is a quite complicated task (see p. 66). In contrast, the procedure suggested by Bronzin is much simpler:

- Compute $F(M)$, i.e. the probability that $\tilde{x}$ exceeds $x=M$. This given by $\frac{(\omega-M)^2}{2\omega^2}$.
- Solve $\frac{\partial P_1}{\partial M}=-F(M)=-\frac{(\omega-M)^2}{2\omega^2}$ for P_1, which is given by the integral $P_1=-\int F(M)dM+c=-\int\frac{(\omega-M)^2}{2\omega^2}dM+c$. The solution is $P_1=\frac{(\omega-M)^3}{6\omega^2}$. Notice that the constant is zero because $P_1(M=\omega)=0$ (see p. 62).

4.4 *Summing up*

The major task in pricing options and other derivatives is to find an appropriate pricing function $f(x)$ which translates future random payoffs into current prices. In Bronzin's own perspective, $f(x)$ is a standard probability density function. However, the mean of his density is not an ordinary, unspecified or subjective expected value, but a market price which can be observed – namely the forward price of the security. His pricing density can thus be regarded as a risk-neutral pricing function – which does not necessarily provide the "correct" statistical probabilities, but prices options in a consistent way with the underlying resp. the forward contract. Based on his equations (16), (17) and (19), Bronzin suggests *three* ways to specify the pricing function $f(x)$:

- Estimate volatilities and probabilities, and fit $F(x)$ by least squares (the derivative $f(x)$ can then be derived).
- Try alternative functional specifications (see Section 5).
- Compute the second derivative $\frac{\partial^2 P_1}{\partial M^2} = f(M)$ for alternative $M = x$ from existing market prices; as shown in equation (7).

5. *Option pricing with specific functional or distributional assumptions*

5.1 *General remarks*

The specification of the pricing density $f(x)$ and the derivation of closed form solutions for option prices is the objective of the 2nd chapter in part II. Bronzin discusses six different functional specifications of $f(x)$ and the implied shape of the density for a given range of x. From a probabilistic point of view, this part of the book seems to be slightly outdated, because the first four "distributions" lack any obvious stochastic foundation. The function $f(x)$ seems to be specified rather ad-hoc, just to produce simple probability shapes for the price deviations from the forward price: a rectangular distribution, a triangular distribution, a parabolic distribution, and an exponential distribution.

This impression particularly emerges if Bachelier's thesis is taken as benchmark, where major attention is given to the modeling of the probability law governing the dynamics of the underlying asset value. This was an extraordinary achievement on its own. In order to be fair about Bronzin's approach, one should be aware of the state of probability theory at the beginning of the last century. As Bernard Bru mentioned in his interview with Murad Taqqu (see Taqqu 2001, p. 5), "probability did not start to gain recognition in France until the 1930's. This was also the case in Germany".

However, the fifth and sixth specification of $f(x)$ are the (normal) law of error (*Fehlergesetz*) and the Bernoulli theorem, or in modern terminology, the normal and binomial distributions. This enables a direct comparison with the Bachelier and the Black-Scholes and Merton models.

For the subsequent discussion it is useful to recall that x denotes the market price of the underlying asset at maturity *minus the forward price*. Bronzin now makes the simplifying assumption that functions $f(x)$ and $f_1(x)$ are symmetric around B, i.e. that $f(x) = f_1(x)$ (p. 55). This assumption makes the expected market price equal to the forward price. At the same time, he is entirely aware that a symmetric probability density is not consistent with the limited liability nature of the underlying "objects": while price increases are potentially unbounded, prices cannot fall below zero[35]. However, he plays this argument down by saying that these (extreme) cases are fairly unlikely, and price variations can be regarded as more or less uniform (*regelmässige*) and generally not substantial (*nicht erhebliche*) oscillations around B. Based on this reasoning, he seems to be very confident about the results being derived from this assumption…[36]

[35] Original text: „*... es könnte ja eine Kurserhöhung in unbeschränktem Masse stattfinden, während offenbar eine Kursemiedrigung höchstens bis zur Wertlosigkeit des Objekts vor sich gehen kann*" (p. 56).
[36] Original text: „ *.... so darf man die gemachte Voraussetzung getrost akzeptieren und ihren Resultaten mit Zuversicht entgegensehen*" (p. 56).

5.2 Option prices under specific distributional assumptions

Exhibit 2 displays the densities derived from the various (six) functional specifications of the terminal price, as well as the implied call option prices (P_1). In what follows, we only add a few comments for each specification – except for the error (i.e. Normal) distribution which constitutes a direct link to the Black-Scholes formula. More details can be found in Hafner/ Zimmermann (2004).

	Density function	*Standard deviation*	*Bronzin's call option price*
Uniform distribution	$f(x)=\frac{1}{2\omega},\ x\in[-\omega;+\omega]$		$P_1=\frac{(\omega-M)^2}{4\omega}$
Triangular distribution	$f(x)=\frac{\omega-x}{\omega^2},\ x\in[-\omega;+\omega]$		$P_1=\frac{(\omega-M)^3}{6\omega^2}$
Parabolic distribution	$f(x)=\frac{3(\omega-x)^2}{2\omega^3},\ x\in[-\omega;+\omega]$		$P_1=\frac{(8\omega-M)^4}{8\omega^3}$
Exponential distribution	$f(x)=ke^{-2kx}$	$\sigma_{\exp}(x)=\frac{1}{2k}$	$P_1=\frac{e^{-2kM}}{4k}$
Error distribution	$f(x)=\frac{h}{\sqrt{\pi}}e^{-h^2x^2}$	$\sigma_{err}(x)=\frac{1}{h\sqrt{2}}$	$P_1=\frac{e^{-M^2h^2}}{2h\sqrt{\pi}}-M\,\psi(hM)$
Bernoulli (binomial) distribution	$\frac{1}{\sqrt{2\pi}}\int_0^{z^*}e^{-\frac{1}{2}z^2}dz+\frac{e^{-z^2}}{\sqrt{2\pi}\sqrt{Bq}}$ $z^*=\frac{x^*}{\sqrt{Bq}},\ \tilde{z}=\frac{\tilde{x}}{\sqrt{Bq}}$	$\sigma_{bin}(x)=\sqrt{qB}$	

Exhibit 1 – Option prices under alternative distributional assumptions, Bronzin (1908)

- Uniform and triangular function distributions

Assuming the same boundaries ω for the uniform and triangular distribution[37], it is interesting to notice that the ATM option prices decrease from one fourth of ω (uniform distribution) to one sixth (triangular). This nicely shows the impact of shifting part of the probability mass (i.e. one eighth on each side of the distribution) from the "tails" to the center of the distribution, or the reverse. To put it differently, the "riskier" uniform density implies an ATM option price which is $\omega/4 \div \omega/6 = 1.5$ times, or respectively 50%, higher than the price implied by the triangular distribution – although only 25% of the probability mass is shifted from the tails to the center.

- Parabolic distribution

Bronzin suggests to use this distribution for modeling extreme values with small probabilities by setting ω sufficiently large (p. 67). Nevertheless, we now assume that ω is the same as in the previous two Sections in order to facilitate comparisons. Since extreme value have again become less likely compared to the triangular distribution, it is not surprising that the value of ATM options is again lower, i.e. it decreases from one sixth of ω to one eighth. The other results are similar and need no further comment.

- Exponential function

The range of x values is unbounded, and rare events with small probabilities can even be handled much easier by this functional specification. The parameter k determines the variability of x – a

[37] This does not keep the standard deviation of the distribution the same, of course.

bigger k reduces the variability. As shown in the next Section, the standard deviation (volatility) of the distribution is given by $\sigma = \frac{1}{2k}$. Then the price of ATM option is straight *half* the volatility! Again, the general option prices separate the impact of the volatility and moneyness in an extremely nice way. The ATM option price under our calibration for k is

$$P\left(k = \frac{3}{2\omega}\right) = \frac{1}{4k} = \frac{1}{4 \times \frac{3}{2\omega}} = \frac{1}{\frac{6}{\omega}} = \frac{\omega}{6}$$

which exceeds the respective option price from the parabolic distribution by $\frac{\omega/6}{\omega/8} - 1 = \frac{1}{3}$, i.e. one third.

- The normal law of error (Normal distribution)

The most exciting specification of $f(x)$ is the law of error (*Fehlergesetz*) defined by $f(x) = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}$ [38]. Unlike the previous specifications of $f(x)$, this is now a direct specification of the probability density. Reasoning that market variations above and below the forward price B can be regarded as deviations from the markets' most favorable outcome, Bronzin suggest to use the law of error as a very reliable law to represent error probabilities[39]. Of course, the density corresponds to a normal distribution with zero mean and a standard deviation of

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[38] The (normal) law of error should not be confused with error *function* which is an integral defined by $erf(x) = 2\int_0^x \frac{1}{\sqrt{\pi}} e^{-t^2} dt$, related to the cumulative standard normal $N\{\}$ by $erf(x) = 2\left[N\{x\sqrt{2}\} - 0.5\right]$.

[39] Original text: „*Indem wir uns also die Marktschwankungen über oder unter* B *gleichsam als Abweichungen von einem vorteilhaftesten Werte vorstellen, werden wir versuchen, denselben die Befolgung des Fehlergesetzes ... vorzuschreiben, welches sich zur Darstellung der Fehlerwahrscheinlichkeiten sehr gut bewährt hat; ...*“ (p. 74).

$\sigma_{err} = \frac{1}{h\sqrt{2}}$. Or alternatively, setting $h = \frac{1}{\sigma\sqrt{2}}$ gives us the normal $N(0, \sigma^2)$.[40]

In order to compare the ATM option price with the previous Section, it is necessary to have equal variances.

(9)... $$\sigma_{err}(x) = \frac{1}{h\sqrt{2}}$$

which shows the standard deviation of the error distribution implied by a specific choice of parameter h. Since h is inversely related to the standard deviation of the distribution, it measures the precision of the observations, and is called *precision modulus;* see Johnson/ Kotz/ Balakrishnan (1994), p. 81.

The relationship between the volatility of the exponential and the error distribution is then given by the equality $2k = h\sqrt{2}$ or

(10)... $$h = k\sqrt{2}.$$

The implied ATM option price is therefore

$$P_{err}\left(h = k\sqrt{2}\right) = \frac{1}{2k\sqrt{2}\sqrt{\pi}} = \frac{1}{k\sqrt{8\pi}} = \frac{1}{5.013 \times k}$$

which is only about 80% of the exponential ATM option price $P_{\exp} = \frac{1}{4k}$. This is not surprising: compared to the exponential distribution, the error (or normal) distribution has more weight around the mean and less around the tails – given the same standard deviation.

[40] As a historical remark, the analytical characterization as well as the terminology related to the "normal" distribution was very mixed until the end of the 19th century; while statisticians like Galton, Lexis, Venn, Edgeworth, and Pearson have occasionally used the expression in the late 19th century, it was adopted by the probabilistic community not earlier than in the 1920s. Stigler (1999), pp. 404-415, provides a detailed analysis of this subject.

- The binomial distribution ("Bernoulli theorem")

While sections 2 through 6 in the 2nd chapter of part II in Bronzin's book are direct specifications of the pricing density $f(x)$, the approach taken in his final section 7 is slightly different. It can be understood as a mere specification of the (inverse) volatility factor h in the error function. The starting point of his analysis is almost identical to the binomial model of Cox/ Ross/ Rubinstein (1979). Assuming that s (consecutive) price movements[41] are governed by "two opposite events" (e.g. market ups and downs) with probability p and q, which can be thought as Bernoulli trials. The expected value of the distribution is sp (or alternatively, sq)[42]. Of course, the events can be scaled arbitrarily by choosing the parameter s appropriately. Therefore, one of the expected values (which one is arbitrary) can be set equal to the forward price, e.g. $B = sp$. The price distribution can then be understood as being generated by cumulative deviations of market events from their most likely outcome, the forward price. The standard deviation of this distribution is $\sqrt{spq} = \sqrt{Bq}$.

- Extensions and generalizations by Gustav Flusser (1911)

We found[43] only one explicit reference to Bronzin's work, which is an article by Gustav Flusser[44] published in the Annual (*Jahresbericht*) of the Trade Academy in Prague. While highly mathematical, the author

[41] Again, there is no reference to a time dimension in Bronzin's approach. In the Cox/ Ross/ Rubinstein (1979) setting, these would be interpreted as consecutive market movements. In the Bronzin setting, the binomial approach is just used to characterize the deviations from the expected (i.e. forward) price.
[42] Original text: „*... so stellen ps resp. qs die wahrscheinlichsten Wiederholungszahlen der betrachteten Ereignisse dar*" (p. 80).
[43] We are extremely grateful to Ernst Juerg Weber from the School of Economics and Commerce, at the University of Western Australia, who called our attention to this paper and made it available to us.
[44] Gustav Flusser studied mathematics and physics, and was a professor at the German and Czech University of Prague. He was also a member of the social-democratic party in the parliament. He starved in the concentration camp of Buchenwald in 1940.

merely extends and generalizes the second part of Bronzin's option pricing formalae for alternative distributions for the underlying price[45]:

- polynomial funtions of n-th degree
- rational algebraic functions
- irrational functions
- goniometric (periodic) functions
- logarithmic functions
- exponential functions.

However, the author does not add original contributions to Bronzin's work, in the sense of general pricing principles or extensions thereof, so there is no need to discuss the paper further here.

5.3 A comparison with the Black-Scholes model

Obviously, the specification of the pricing function in the previous section is particularly interesting, because it promises a direct link to the celebrated Black-Scholes model[46]. As seen before, setting $h = \frac{1}{\sigma\sqrt{2}}$ in the error function generates the normal distribution. The problem is, however, that the Black-Scholes model assumes a normal distribution for the *log*-prices, while Bronzin makes this assumption for the price *level* itself. In terms of the underlying stochastic processes, Bronzin's distribution can be regarded[47] as the result of an arithmetic

[45] The author motivaties the paper as follows (original text): *Die vorliegende Arbeit will auf Grund der Untersuchungen Bronzin's die Höhe der Prämie bei den verschiedenen Formen, welche die Börselage annehmen kann, bestimmen, die von ihm gewählte endliche und stetige Funktion der Kursschwankungen* $f(x)$ *auf allgemeine Basis stellen und derselben die Form der .... Funktion erteilen.* (p. 1)

[46] We adopt the common terminology in using „Black-Scholes“ for the models developed by Black/ Scholes (1973) and Merton (1973).

[47] There is however no reference to a specific stochastic process in Bronzin's text.

Wiener process, while the Black-Scholes model relies on a geometric Wiener process.

Since there is an immediate link between the two processes, why not interpreting Bronzin's price levels as log-prices? This is, however, not adequate in the option pricing framework because the value of options is a function of the payoff emerging from the (positive) difference between settlement *price* and exercise *price* of the option, not their logarithms. In this respect, the approach of Bronzin is the same as the one of Bachelier.

We show how to rewrite the Bronzin's equation to get the Black-Scholes formula. For this purpose, we replace

(12)... $\tilde{x} - M = \left[\tilde{S}_T - B\right] - \left[K - B\right] = \tilde{S}_T - K$,

and assume that $\tilde{S}_T$ is *log*normally distributed, which we write in terms of the standard normal $\tilde{z}$ as

(13a)... $\tilde{S}_T = S_t e^{\mu(T-t)+\sigma\tilde{z}\sqrt{T-t}}$, with $\mu = \frac{E\left[\ln\left(\frac{S_T}{S_t}\right)\right]}{T-t}, \sigma^2 = \frac{Var\left[\ln\left(\frac{S_T}{S_t}\right)\right]}{T-t}$.

Adapting the risk-neutral valuation approach of Cox/ Ross (1976), the drift of the log stock price changes can be replaced by $\mu = r - \frac{1}{2}\sigma^2$. In order to facilitate the comparison with Bronzin, we subsequently assume an interest rate of zero and one time unit to maturity, $T - t = 1$ (e.g. one year if volatility is measured in annual terms). The forward price is then equal to the current stock price, implying

(13b)... $\tilde{S}_T = Be^{-\frac{1}{2}\sigma^2 + \sigma\tilde{z}}$.

The Black-Scholes valuation equation can then be written as

$$(14)\ldots \qquad P_1 = \int_{-z_2}^{\infty} \left(\underbrace{Be^{-\frac{1}{2}\sigma^2 + \sigma\tilde{z}}}_{S_T} - K \right) N'(z)dz,$$

where the remaining task is to adjust the lower integration boundary, here denoted by $-z_2$ in anticipation of the Black-Scholes model. For this task, we just have to transform the probability range of the normal $\tilde{x}$, $pr(\tilde{x} > M)$, to a new range $pr(\tilde{S}_T > K)$ expressed relative to the standard normal density $N'(z)$. Notice that $pr(\tilde{S}_T > K)$ is equal to $pr(\ln S_T > \ln K)$, and that $\ln(S_T)$ is normally distributed with mean $\ln S_0 - \frac{1}{2}\sigma^2 = \ln B - \frac{1}{2}\sigma^2$ and standard deviation σ. Thus we can standardize both sides of the inequality $pr(\ln S_T > \ln K)$ to get

$$pr\left(\underbrace{\frac{\ln \tilde{S}_T - \left[\ln B - \frac{1}{2}\sigma^2\right]}{\sigma}}_{\tilde{z}} > \frac{\ln K - \left[\ln B - \frac{1}{2}\sigma^2\right]}{\sigma} \right)$$

where $\tilde{z}$ is the standard normal. The expression on the r.h.s. can be written as

$$(15)\ldots \quad \frac{\ln K - \left[\ln B - \frac{1}{2}\sigma^2\right]}{\sigma} = -\frac{\ln B - \ln K - \frac{1}{2}\sigma^2}{\sigma} = -\frac{\ln\frac{B}{K} - \frac{1}{2}\sigma^2}{\sigma} \equiv -z_2$$

which is exactly the Black-Scholes boundary typically expressed as

$$pr(\tilde{z} > -z_2) = pr(\tilde{z} < z_2) \equiv N(z_2).$$

Summing up, we have shown that the Bronzin equation (11) can be easily transformed to the Black-Scholes model if the stock price $B + \tilde{x} = \tilde{S}_T$ is specified as a lognormal instead of a normal variable and the integration boundary is adjusted correspondingly. Thus, the pricing relationship looking most similarly to the Bronzin equation is

$$P_1 = \int_K^\infty \left(\tilde{S}_T - K\right) L'\left(S_T\right) dS_T = \int_{-z_2}^\infty \left(Be^{-\frac{1}{2}\sigma^2 + \sigma\tilde{z}} - K\right) N'(z) dz, \quad z_2 = \frac{\ln\frac{B}{K} - \frac{1}{2}\sigma^2}{\sigma}$$

The explicit solution using the standard Black-Scholes procedure in transforming the integral[48] is

$$P_1 = B\, N\left\{z_2 + \sigma\sqrt{T-t}\right\} - K\, N\left\{z_2\right\}$$

or re-adapting time and interest,

$$P_1 = Be^{-r(T-t)}\, N\left\{z_2 + \sigma\sqrt{T-t}\right\} - Ke^{-r(T-t)}\, N\left\{z_2\right\}$$

where $Be^{-r(T-t)} = S_t$ can also be written as the current stock price. This derivation shows that Bronzin's valution equation (11) is fully consistent with the Black-Scholes, and, respectively, the Black (1976) forward price based valuation models. It is also a risk-neutral valuation approach – he makes no assumptions on preferences or expected values – simply because the option price relies on the forward price and zero (random) deviations from there.

[48] see e.g. James (2003), pp. 299-309.

6. *Option pricing in historical perspective*

Judgements about scientific originality are always difficult with a delay of a century, in a field which has progressed so rapidly as option pricing, and where statistical and stochastic methods are used which were hardly developed at this time. It is even questionable whether scientific originality is a fair criterion to apply – because nothing is known about its purpose or target audience. Given that he published it as a "professor", and given that he has published a textbook on actuarial theory for beginners two years before (Bronzin 1906), it may well be that he regarded his option theory as a textbook, or a mixture between textbook and scientific monograph. Finally, Bronzin did definitively not overstate his own contribution – he even understates it by regularly talking about his "booklet" (*Werkchen*) when referring to it[49]. Why he was talking about his textbook as a "booklet" is an open question: Was it, because it was not good for his reputation as an academic to write about financial mathematics – or worse, on a topic typically associated with speculation? Was it because the subject was too far away from his profession as a professor for arithmetic. We do not know. Further research has to be done.

The originality in the field of option pricing is difficult to assess anyway. Who deserves proper credit for the Black-Scholes model? The early Samuelson (1965) paper contains the essential equation[50]. Even more puzzling is a footnote in the Black-Scholes paper (p. 461) where the authors acknowledge a comment by Robert Merton suggesting that if the option hedge is maintained continuously over time, the return on the hedged position becomes certain. But it is the notion of the riskless hedge which makes the essential difference between Black/ Scholes and the earlier Samuelson and Merton/ Samuelson

[49] The German word is actually a funny combination of *Work* which means, in an academic setting, a substantial contribution, while the ending ...*chen* is a strong diminutive.
[50] Or to use Samuelson's own wording: "Yes, I had the equation, but "they" got the formula…"; see Geman (2002).

models[51], [52]! Surprisingly enough that Merton was kind enough to delay publication of his (accepted) 1973 paper until Black/ Scholes got theirs accepted[53].

An open question is to what other publications Bronzin is referring to: He surely knew the most important publications in German language about probability and options. Options ("Prämiengeschäfte") were well known instruments at this time at the stock exchanges in the German spoken part of Europe. Their were many different forms. And at least about the legal aspect of the options different books containing financial transactions have been published.[54] But the mathematical background of options didn't seemto be an issue. Another question is, whether Bronzin knew about Bachelier's work. *Honni soit qui mal y pense* … - but extensive quoting was not the game at the time anyway. Bachelier did not quote any of the earlier (but admittedly, non mathematical) books on option valuation either. For example, the book of Regnault (1863) was widely used and contains the notion of random walk, the Gaussian distribution, the role of volatility in pricing options, including the square-root formula[55]. According to Whelan (2002) who refers to a paper by Émile

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[51] To be precise, the notion of a "near" risk-less hedge strategy can also be found in the Samuelson and Samuelson/ Merton papers. Samuelson (1965) analyses the relationship between the expected return on the option (warrant), β, and the underlying stock, α, and argued that the difference "cannot become too large. If $\beta > \alpha$ […] hedging will stand to yield a sure-thing positive net capital gain (commissions and interest charges on capital aside!)"(p. 31). Samuelson/ Merton (1969) extend the earlier model and derive a "probability-cum-utility" function Q (see p. 19), which serves as a new probability measure (in today's terminology) to compute option prices. They show that under this new measure (or utility function), all securities earn the riskless rate; they explicitly write $\alpha_Q = \beta_Q = r$ to stress this point (see p. 26, equations 20 and 21 and the subsequent comments). Although Merton and Samuelson recognized the possibility of a (near) risk-less hedge and a risk-neutral valuation approach, they were not fully aware of the consequences of their findings.

[52] Black (1988) gives proper credit to Robert Merton: "Bob gave us that [arbitrage] argument. It should probably be called the Black-Merton-Scholes paper".

[53] Bernstein (1992) and Black (1989) provide interesting details about the birth of the Black-Scholes formula.

[54] Eg. Leitner (1920) p 624

[55] The argument is derived from a funny analogy: He considers the mean (or fair) value of an asset as the *center of a circle, and* every point within the circle represents a possible future price. The radius describes the standard deviation. He then assumes that, as time elapses, the range of possible stock prices as represented by the area within the circle increases proportionally. This implies that the radius (i.e. the standard

Dormoy published in 1873, French actuaries had a reasonable idea to price options well before Bachelier's thesis, although a clear mathematical framework was missing. Einstein in his Brownian motion paper (1905) did not quote Bachelier's thesis, but it is a generally accepted view that he did not know it. Distribution of knowledge seems to have been pretty slow at this time, particularly between different fields of research, and across different languages. And again, extensive references were simply not common in natural sciences (e.g. Einstein's paper contains a single reference to another author).

If Einstein did not know Bachelier's thesis, it is even less likely that Bronzin knew it; based on what we know from his other work (Bronzin 1906), his general mathematical interests were also quite apart from those of Bachelier. But after all, we do not know what Bronzin knew about other's work on option pricing, but the question is not so relevant either, because there are sufficiently many innovative elements in his treatise. It is also surprising that (almost) no references are found *on* his work, particularly in the German literature. Although it is generally claimed that Bachelier's thesis was lost until the Savage-Samuelson rediscovery, it was at least quoted since 1908 in several editions of a French actuarial textbook by Alfred Barriol (at least until 1925).

Bronzin's book had a similar recognition. As stated earlier, it was mentioned in Leitner's book about banking in Germany, published in four editions. And with Bronzin's more pragmatic pricing approach, it is difficult to understand why the seeds for another, more scientific understanding of option pricing did not develop, or the formulas did not get immediate practical attention. At least, Bronzin was not a doctoral candidate as Bachelier, but (apparently) a distinguished professor; moreover, the flourishing insurance industry in Trieste should have had an active commercial interest in his re-

deviation) increases with the square root of time. A detailed analysis of Regnault's contribution is given in several papers by Jovanovic and Le Gall; see e.g. Jovanovic/ Le Gall (2001).

search. While Poincaré's reservation on Bachelier's thesis is, at least, limited to his "queer" subject[56] and can, somehow, be understood from a purely academic point of view, it is more difficult to understand why a reviewer of Bronzin's book, in 1910, commented that "it can hardly be assumed that the results will attain a particularly practical value"…[57]. It however evidences Hans Bühlmann's and Shane Whelan's[58] claim that the contribution of actuaries to financial economics is generally underestimated (see Whelan 2002 for detailed references).

[56] Poincaré was the main adviser of Bachelier's thesis; he writes in his report: «*Le sujet choisi par M. Bachelier s'éloigne un peu de ceux qui sont habituellement traités par nos candidats*»; Taqqu (2001), Appendix.

[57] Orig. text: „*Es ist kaum anzunehmen, daß die bezüglichen Resultate einen besonderen praktischen Wert erlangen können, wie ja übrigens auch der Verfasser selbst andeutet*". The last part of the sentence ("... which is also noticed by the author") is simply not true. The book review was published by an anonymous author in the *Monatshefte für Mathematik und Physik* (Volume 21; mit Unterstützung des Hohen K. K. Ministeriums für Kultus und Unterricht, Wien, Verlag des Mathematischen Seminars der Universität Wien).

[58] See Whelan (2002) for detailed references.

7. *Beyond Finance: The probabilistic and historical background of Bronzin's work*

Applying probabilistic models to financial problems was common in actuarial science, particularly life insurance, at the end of the 19th century, but not in areas related to speculation, financial markets, or derivative contracts. In this respect, the work of Bronzin as well as of Bachelier marked a substantial breakthrough.

It is not easy to identify the intellectual foundations of Bachelier's and Bronzin's works. It could be found in the "probabilistic" revolution[59] which took place in physics, and to some extent in economics, in the second part of the 19th century. In this context, it may regarded as another amazing parallel between of the lives and achievements of Bachelier and Bronzin that they were both students in an environment of theoreticians in search of new analytical tools for getting a deeper and new understanding of the intrinsic structure of the world: entropy and probability. As noted earlier, Bachelier submitted his thesis to Henri Poincaré, and Bronzin took courses and seminars with Ludwig Boltzmann at the Technical University of Vienna[60]. Both, Poincaré and Boltzmann, building on the foundations laid by Maxwell, laid the mathematical foundations of modern physics – although their approach was different[61]. Is there a relationship to the work of Bachelier and Bronzin?

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[59] This term is borrowed from Krüger/ Gigerenzer/ Morgan (1987).

[60] Based on our communication with his son, Andrea Bronzin, who also showed us testimonies signed by L. Boltzmann.

[61] An excellent description of this topic can be found in chapter 14 (Volume 2) in Krüger/ Gigerenzer/ Morgan (1987), contributed by Jan von Plato.

7.1 Probabilistic modelling in physics and finance

Maxwell's achievement was a statistical formulation of the kinetic theory of gas in the 60s of the 19th century. According to kinetic theory, heat is due to the random movement of atoms and molecules, so it looks much like kinetic energy. In contrast to other forms of energy, however, these movements cannot be observed or predicted, while other energies result from orderly movements of particles. Maxwell argued, although random in nature, the velocity of molecules can be described by mathematical functions – derived from the laws of probability.

It is the same reasoning which is found in the introductory sections of Bachelier's and Bronzin's writings: They both argue that although speculative markets (prices) behave in a completely random and unpredictable way, this does not prevent, but rather motivate, the use of mathematical – probabilistic – tools. This is reflected by the following quotes:

> *« Si le marché, en effet, ne prévoit pas les mouvements, il les considère comme étant plus ou moins probables, et cette probabilité peut s'évaluer mathématiquement » ;* Bachelier (1900), pp. 21-22.

> *«…ebenso klar ist es aber auch, dass sich die Ursachen dieser Schwankungen und somit die Gesetze, denen sie folgen sollten, jeder Rechnung entziehen. Bei dieser Lage der Dinge werden wir also höchstens von der Wahrscheinlichkeit einer bestimmten Schwankung x sprechen können, und zwar ohne hiefür einen näher definierten, begründeten mathematischen Ausdruck zu besitzen; wir werden uns vielmehr mit der Einführung einer unbekannten Funktion $f(x)$ begnügen müssen…";* Bronzin (1908), pp. 39-40.

This marked a fundamental change in the perception of risk in the context of financial securities.

Back to Poincaré and Boltzmann – things becom slightly more complicated. Their approach to model the unpredictability, irreversibility, or chaotic behavior of dynamical systems was quite different and created much controvery. It was not clear how to reconcile probabilistic and statistical laws with the mechanical laws of Newtonian physics.

Boltzmann addressed the problem by proofing the irreversibility of macroscopic systems through kinetic gas theory – which is, after all, a purely mechanic, deterministic point of view: While any *single* molecule obeys the classical rules of reversible mechanics, for a large *collection* of particles, he claimed, that the *laws of statistics* imply irreversibility and force the second Law to hold. From any arbitrary initial distribution of molecular velocities, molecular collisions always bring the gas to an equilibrium distribution (as characterized by Maxwell). In a series of famous papers included as chapter 2 and 3 in Boltzmann 2000) he showed that, for non-equilibrium states, the entropy is proportional to the logarithm of the probability of the specific state. The system is stable, or in thermal equilibrium, if entropy reaches its maximum – and hence, the associated probability. So, maximum entropy (disorder) is the most likely – and hence: equilibrium - state in a thermodynamic system. In short: Boltzmann recognized "how intimately the second Law is connected to the theory of probability and that the impossibility of an uncompensated decrease of entropy seems to be reduced to an improbability"[62].

This theorem is widely regarded as the foundation of statistical mechanics, by describing a thermodynamic system using the statistical behavior of its constituents: It relates a *microscopic property* of the system (the number or probabilities of states) to one of its *thermodynamic properties* (the entropy).[63] However, he was heavility criticized, because, after all, it was a purely *meachanical* proof of the second law of Thermodynamics: he used "laws of probability" to bridge the con-

[62] See Klein (1973), p.73.
[63] See Fischer (1990), p 167.

flict between *macroscopic* (thermodynamic) irreversibility and *microscopic* (mechanical) *reversibility* of molecular motions.

It is therefore not surprising that Boltzmann's probabilistic interpretation of entropy was not accepted by all researchers at that time without reservation, and created much quarrel, controversy, and polemic. While Boltzmann (and Clausius) insisted on a strictly mechanical interpretation of the second Law, Maxwell still claimed the statistical character of the Law. A major objection came in 1896 from one of Planck's assistants in Berlin (E. Zermelo) which is particularly interesting in our context – it is the place where Poincaré enters the scene. Zermelo referred to a mathematical theorem published by Poincaré in 1893 which implies that any spatially bounded, mechanical system ultimately returns to a state sufficiently close to its initial state after a sufficiently long time interval. This was inconsistent with Boltzmann's theorem (and a kinetic theory of gas in general). If the validity of mechanical laws is assumed for thermodynamic processes on a microscopic level, entropy cannot increase monotonically, and irreversible processes are impossible: hence, the world is not a mechanical system!

It is amazing to see how notable scientists resisted to "swap the solid ground of the laws of thermodynamics – the product of a century of careful experimental verification – for the ephemeral world of statistics and chance" (Haw 2005). Boltzman himself considered kinetic theory as a purely mechanical analogy; after all, nobody had ever physically observed the particles kinetic theory was all about.

This was actually done by Albert Einstein's by investigating the Brownian motion, i.e. the old observation from Robert Brown in the early 19th century, that small particles in a liquid were in constant motion, carrying out a chaotic "dance" – not being caused by any external influence. Was this a violation of the second Law on the level of single particles? Einstein was able to prove that liquids are really made of atoms, and experiments moreover demonstrated that the movement of the Brownian particles were perfectly in line Boltzmann's kinetic gas theory! Thus, Einstein successfully integrated the thermodynamics of liquids with Boltzmann's interpretation of the second Law with statistical mechanics – Boltzmanns' vision at the end of his (1877)-paper proved to be right: He claimed, that it is very likely that his theory is not limited to gases, but represents a natural law applicable to e.g. liquids as well, although the mathematical difficulties of this generalization appeared "extraordinary" to him:

> *"Es kann daher als wahrscheinlich bezeichnet warden, dass die Gültigkeit der von mir entwickelten Sätze nicht bloss auf Gase beschränkt ist, sondern dass dieselben ein allgemeines, auch auf … und tropfbar-flüssige Körper anwendbares Naturgesetz darstellen, wenngleich eine exakte mathematische Behandlung aller dieser Fälle dermalen noch auf aussergewöhnliche Schwierigkeiten zu stossen scheint"* (Boltzmann 1877; 2000, p. 196).

Einstein formulated a theory of Brownian motion in terms of a differential equation – the celebrated diffusion equation (Einstein 1905). But again – while Einstein could easily live with statistical concepts in the context of atoms, he was never in favour of a statistical or probabilistic interpretation of quantum mechanics ("God does not play dice"…). Today, much of the controversy whether a deterministic or a stochastic system is needed to cause the irreversibility of macroscopic processes is alleviated – chaos theory has established as a powerful mathematical intermediary. Poincaré was one of the pioneers in this field – but nevertheless, Boltzmann was aware as well that the dynamic properties of a thermodynamic system depend crucially on the initial state of the system, and prediction becomes impossible [65].

What has all this to do with finance? A lot – because it is well known that Einstein's mathematical treatment of the Brownian mo-

[65] This statement originates from a reply to one of Zermelo's criticisms; see Fischer (1990), p. 174.

tion was pioneered by Bachelier. The surprising fact is, however, that Bachelier wrote his thesis under supervision of Henri Poincaré, whose sympathy with the probabilistic modelling of dynamic systems was, as discussed before, limited. It is in fact amazing how strong Bachelier's belief was in the power of probability theory - Delbean/ Schachermayer (2001) even call it "mystic". This is best reflected in the concluding statement of his thesis:

> *« Si, à l'égard de plusieurs questions traitées dans cette étude, j'ai comparé les résultats de l'observation à ceux de la théorie, ce n'était pas pour vérifier des formules établies par les méthodes mathématiques, mais pour montrer seulement que le marché, é son insu, obéit à une loi qui le domine: la loi de la probabilité. »* Bachelier (1900), p. 86.

Maybe, this exuberant commitment to probability was not too beneficial for the overall evaluation of the thesis by his advisor, Poincaré! After all, "it must be said that Poincaré was very doubtful that probability could be applied to anything in real life …" (Taqqu 2001, p. 9) which was fundamentally different from Bachelier's view and ambition.

In any case, Bacheliers' approach would have emerged more naturally from Boltzmann's statistical mechanics. The similarity of the theoretical reasoning is most evident if one compares the first page of Bachelier's thesis, where he describes the motivation and adequacy of probability theory for characterizing stock price movements, with the setup of Boltzmann's (1877) kinetic gas theory. The uncountable determinants of stock prices, their interaction and expectation seem to have a similar (or the same?) role with respect to the unpredictability (or maximum chaos) of the system as the collision of innumerable small molecules and the second law of thermodynamics.

Was thermodynamics ever applied to economic modelling? While not in a probabilistic setting, Vilfredo Pareto (1900) made an analogy with the second Law in discussing the redistribution of wealth between individuals by changing the conditions of free competition. He claims that this process necessarily leads to a corrosion of welath – and attributes to this "theorem" the same (or "analogeous") role as the second Law in physics:

> *"Man kann den Reichtum von bestimmten Individuen auf andere übertragen, indem man die Bedingungen der freien Konkurrenz abändert, sei es in Bezug auf die Produktionskoeffizienten, sei es in Bezug auf die Umwandlung der Ersparnisse in Kapitalien. Diese Übertragung von Reichtum ist notwendigerweise mit einer Zerstörung von Reichtum verbunden … Dieses Theorem spielt in der Wirtschaftslehre eine analoge Rolle wie das zweite Prinzip der Thermodynamik." (*Pareto 1900, p. 1119).

But we are not aware of entropy-based foundations of economic systems or financial markets around the turn of the century. Was there a probabilistic revolution in economics at all?[67]

Unfortunately, Bronzin being an admiring student of Boltzmann, did not use any element of statistical mechanics for modelling price processes or their distribution – which is a surprising fact indeed. Rather, his approach was more in the probabilistic tradition of actuarial science.

7.2 *Probability in actuarial science and the treating of market risks*

A lot has been written about the long tradition of probabilistic modelling in actuarial science, and there is no need to replicate the history here. Also needless to say that actuarial science played a pivotal role for the expansion of the insurance sector as the driving force behind the economic growth and industrialization in the 19th century.[68] By

[67] See Krüger/ Gigerenzer/ Morgan (1987), Volume 2, Chapter 6, about this point.

[68] It is interesting to see how nation-building and the development of the old-age-pension-system paralleled each other. For example, Bismarck installed the state-sponsored old-age-pension-system with the intention to create a conservative attitude by the workers. LOTH (1996), p. 68, quotes Bismarck: The pension system

reading actuarial textbooks and monographs published in German, towards the end of the 19th century, three (related) features are apparent:

First, we observe a more rigorous probabilistic treatment of the key concepts of insurance mathematics – the emergence of elements of a formal "risk theory". A good example is an encyclopaedia article on "insurance mathematics" by Georg Bohlmann (1900) containing an axiomatic treatment of probability containing many elements of Kolmogorov's famous treatment 33 years later. This resulted from the insight that the insurance business needed a more solid, *scientific basis* for calculating risks, covering potential losses and determining adequate premiums[69]. Also, there was an increasing demand for a precise, probability-based terminology of the key actuarial terms; this is reflected in the following statement (related to a book review):

> *"Die Begriffe: Nettoprämie, Jahresrisico, Prämienreserve u.s.w. sind uns geläufig, wie sie sie erlernt, wir operiren mit ihnen, ohne zu untersuchen, ob sie ausreichend oder gar präcise definirt sind. Werden diese Begriffe …. vor der eingehenden Kritik Stand halten können? …. Ich glaube es aber mit nichten"*[70].

A *second* observation is the increasing analogy between the nature of insurance contracts and "games of chance" (*Zufallsspiele*). An early although non-mathematical characterization of this kind is Herrmann (1869), and a rigorous mathematical treatment is Hausdorff (1897); both authors characterize insurance contracts as special forms of games of chance[71]. Hausdorff's treatise is particularly revealing; he

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was created „ … *[um] in der grossen Masse der Besitzlosen die konservative Gesinnung (zu) erzeugen, welche das Gefühl der Pensionsberechtigung mit sich bringt*."

[69] Assicurazioni Generali (in Trieste) was apparently very proud to publish the actuarial foundations of its life business in 1905, elaborated by Vitale Laudi and Wilhelm Lazarus over many years, as an opulent monograph. But ironically, in 1907, Generali changed their foundations of its life business and re-adopted the generally used formula of Gompertz-Makeham (see: "*Die Jahrhundertfeier der Assicurazioni Generali,* Trieste 1931, p. 99).

[70] Julius Altenburger in a book review on Karl Wagner:„Das Problem vom Risico in der Lebensversicherunmg" (Jena 1898), published in: *Oesterreichische Revue. Organ für Assecuranz und Volkswirthschaft 23*, No. 38, 19 September 1898.

[71] The term "games of chance" (*Zufallsspiele*) is already used by the physiologist, logician, philosopher and mathematician Johannes Von Kries (1886), chap. 3 and 7, although not in a rigorous mathematical setting.

analyzes different types of (what we would call nowadays) financial contracts, their expected loss and profit for various parties. He also analyzes the impact of various amortization or redemption schedules on optimal call policies and bond prices (such as for callable bonds, lottery bonds, premium bonds)[72].

This directly leads to the *third* observation, namely the increasing – although still quite limited - perception of market risk – as opposed to the (traditional) actuarial risk[73]. The growing perception of market risk was caused, among other things, by substantial and permanent deviations of market interest rates from their actuarial (fixed) level, as well as by the substantial losses insurance companies suffered during the stock market crash in the 70s. Companies were forced to hold special reserves[74] (*Kursschwankungsreserven*). Although the analytical methods were quite advanced, the treatment and economic understanding of market risk was quite limited. Even Emanuel Czuber (1910), a renowned Professor at the Technische Hochschule in Vienna spezializing in insurance mathematics, was pessimistic whether a formal "risk theory" could be helpful for managing market risk:

> *"Als wesentlichste dieser Aufgaben [der Risikotheorie] wird … die rechnungsmässige Bestimmung desjenigen Fonds hingestellt, der … notwendig ist, um das Unternehmen gegen die Folgen eines eventuellen Verlustes aus Abweichungen von den Rechnungselementen mit einem vorgegebenen Wahrscheinlichkeitsgrade zu schützen".*

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[72] The treatise also contains a lucid discussion on the distinction between aggregate and average risk of games, i.e. the distinction between adding and sub-dividing risks. Samuelson (1963) is typically credited for this clarification.

[73] The insignificant perception of market risk before the 70s is, for instance, reflected in Herrmann's (1869) treatise of insurance companies, devoting 4 lines (!) to interest rate uncertainty, by stating that the problem can be handled simply by choosing a sufficiently low actuarial rate in the computation of premia.

[74] Between 1878 and 1884, Assicurazioni Generali increased these newly created reserves ("*Reserve für die Coursschwankungen der Werthpapiere*") from 43'000 to 845'000 Kronen, or in relation to the book value of equity, from 1% to 16%; source: *Jahresbericht der Generali-Versicherungsgesellschaft für 1884*, Trieste 1885, p. 6.

In simple terms: risk theory is about computing VaR- (value-at-risk) based reserves to cover the risks from inadequate actuarial assumptions (e.g. interest rates). But Czuber claims that risk theory is not applicable to interest rate risk, because

> *"… [die Risikotheorie] ruht auf dem Boden der zufälligen Ereignisse…. Die Änderungen des Zinsfusses …. tragen aber nicht den Charakter des Zufälligen an sich, das Systematische waltet hier vor."* (Czuber 1910a, p. 411).

i.e. interest rates do not behave randomly!? Even if this would be correct – what about other market risks? Indeed, the same author argues elsewhere[75], that past asset returns (*Verzinsung*) behave so randomly (*unregelmässig*) that they cannot be used to predict future returns:

> *"Aus den Erfahrungen kann wohl ein Bild darüber gewonnen werden, wie sich die Verzinsung der verschiedenen Anlagewerte in der Vergangenheit gestaltet hat; bei dem unregelmässigen Charakter der Variationen, die oft durch lange Zeiträume unmerklich vor sich gehen, um dann plötzlich ein starkes Tempo einzuschlagen, lässt sich ein begründeter Schluss auf die Zukunft schwer ziehen."* (Czuber 1910a, p. 233).

Obviously, there was no consistent picture about market risks and their probabilistic (stochastic) modelling – which is quite representative for the actuarial literature at this time. Therefore, Bronzin's (1908) contribution constituted a substantial step forward. Armed with standard tools from probability theory, he took the challenge to price the specific type(s) of derivative contracts extensively discussed in the earlier sections of this paper.

7.3 Bronzin's interests and academic work

Why was Bronzin interested in probability theory? Why was he interested in derivative (option) contracts? We have only partial an-

[75] by discussing the difficulties in determining an adequate, long-term actuarial interest rate (or average return level).

swers, or hypotheses, to these questions[76]. In accordance with his son Andrea Bronzin we suggest that Vinzenz Bronzin wrote his (1908) book for educational purposes.[77] This seems to be true for all his earlier publications (e.g. 1904, 1906, 1908), which grew out of subjects of his lectures at the Accademia di Commercio e Nautico in Trieste, where he was a professor for "political and commercial arithmetic". Both fields were part of the mathematical curriculum and also included actuarial science and probability theory. *Political arithmetic* was strongly focused on the needs of the insurance companies.[78] *Commercial arithmetic* was more accomplished to the needs of the banking industry and international orientated trading companies.[79] At this time, it was a well established tradition among professors to publish books about the topics they covered in their lectures[80].

The first publication of Bronzin which is documented in his own curriculum is a short article entitled "*Arbitrage*" in a German journal for commercial education (Bronzin 1904)[81]. The paper is about characterizing relative price ratios of goods across different currencies and associated trading (arbitrage) strategies. While interesting *per se*, it is unfortunately not directly related to the "arbitrage valuation principle" of derivatives valuation – which

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[76] Although we had the opportunity for extensive talks with his (92 old) son, Andrea Bronzin, many questions remain open because Andrea was born *after* the time period most relevant for our research (1900-1910).

[77] From a letter dated 17/01/2005: "*Mio padre ha scritto la teoria delle operazioni a premio perché attinenti al suo insegnamento presso l'Accademia di Commercio di Trieste ed alter Accademie di Commercio austriache.*"

[78] The program at the Accademia inculded: „*Elementi di calcolo di probabilità (probabilità assoluta, relativa, composta. Probabilità rispetto alla vita dell'uomo. Durata probabile della vita. Aspettativa matematica e posta e posta legittima nei giuochi di sorte).*" Source: (1917), pp. 163-164.

[79] For example: „*Arbitraggio di divise, effetti, valuti e di riporto. Borse. Affari commerciali secondo le norme di Borsa in merci ed effetti. Arrangement... Spiegazione delle quotazioni di divisen e valute sulle piazze commerciali d'oltremare più importanti per l'importazione ed esportazione europea.*" Source: Subak (1917), p. 164.

[80] See Subak (1917), pp. 257ff, and Piccoli (1882).

[81] We found only one reference to this paper, in Subak (1917), p. 274. The aim of the journal was to publish critical and original surveys on subjects relevant for educational purposes, contributed by the leading scholars in the field ("*Die 'Monatsschrift für Handels- und Sozialwissenschaft' berichtet über alle das Gebiet .... (des) Unterrichtswesen betreffenden Fragen in kritisch zusammengefassten Originalartikeln von ersten Fachleuten*"); Source: *Monatsschrift für Handels- und Sozialwissenschaft 12* (15 December 1904), pp. 356-360.

Bronzin, ironically, uses in his option pricing booklet, however whithout using this term.

Bronzins second publication (Bronzin 1906) is a monograph on Political Arithmetic (*Lehrbuch der politischen Arithmetik*); it was approved by the minstery of education as an official textbook to be used at the commercial schools and academies in the Empire[82]. Bronzin had not – in contrast to many of his collegues at the Accademia – published extensively. It is therefore more than surprising, if not strange, that he did not quote his (1908) option pricing piece in a publication *(a festschrift)* released for the centenary of the school (subsequently quoted as Subak 1917)! Had it become such a "queer" subject in the meantime? As shown in the last section, it was indeed unusual to apply probability theory to speculation and financial securities pricing in these times, but why should he suppress his major scientific contribution he had produced so far? Was the subject too complicated for the target audience, or did he get frustrating responses?

It is true that gambling, speculation, or trading with derivatives did not enjoy a major popularity around this time[84]. In the last decade of the 19th century, derivatives were more and more blamed to cause exuberant market movements and to be socially harmful. Furthermore, in 1901, a court of justice accepted the "gambling" argument (*Spiel und Wette*) in a legal case in Vienna. Thereafter, forward trading declined and got more and more unimportant.[85] At the rather small stock-exchange of Trieste, premium contracts have not been traded at all during these years.[86] But was this a sufficient reason for Bronzin to suppress this publication? After all, our overall impression is that

[82] This is reflected in the sub-title of the book: „… *zum Gebrache an Höheren Handelsschulen (Handelsakademien) sowie zum Selbstunterricht*".
[84] See Stillich (1909), pp. 1-18, pp. 181-227, for a representative discussion of these issues at that time.
[85] Schmitt (2003), p. 145
[86] Archivio dello stato di Trieste, atto "*Listino Ufficiale della Borsa di Trieste*" from 1900 to 1910.

Bronzin's interest in derivatives (and finance in general) was predominantly on the theoretical side.

Writing books must have been hard work for Bronzin anyway. Beside his academic position, Bronzin was nominated director of the Accademia in 1909, but he was not yet able to accept the nomination, because he was suffering from a strong nervousness, apparently caused by his efforts of writing the two books ("in forte nervosità" because of "compilazione e publicazione di libri matematici").[88] One year later he was offered the same position again, and he then accepted. He resigned in 1937 at the age of 65. His major achievement as a director of the Accademia was seen in his ability to guide the school through a time of big political turbulences before, during and after the first world war. He still preserved a great reputation as mathematician. Moreover, at least during his study years in Vienna, he had the reputation of being a successful gambler.[89] Combining mathematics with gambling seem to have been a perfect fit to write his option pricing theory. Interestingly, no consulting activities are known or documented. He was several times asked to join insurance companies but preferred to stay in academia.[90]

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[88] Archivio dello stato di Trieste, atto *Accademia di commercio e nautica in Trieste*, b 101 e regg 273, 1909, AA 345/09, from the 31.07.1909. In August 1909, also one of his beloved daughters died.
[89] Orbitrary of his nephew, Angelo Bronzin.
[90] Letter as of 30 December 2004 from Arcadio Ogrin, summarizing a conversation with Andrea Bronzin.
[92] De Tuoni (1925).

Vinzenz (later: Vincenzo) Bronzin was born in Rovigno (today: Rovinj), a small town on the peninsula of Istria (Croatia), on 4th May 1872, and died in Trieste on the 20th December 1970 at age 98. He was the son of a Slovenian commandant of a sailing-ship. After completing the gymnasium (high school) in Capodistria, a town on Istria, he became a student in engineering at the University of Polytechnics in Vienna, where he made his exams after an enrolment of two years. He then studied mathematics and paedagogics at the University of Vienna, and at the same time, he took courses for military officers in Graz.

In his obituary, his nephew Angelo Bronzin reports that he was a well known gambler and a champion in fencing during his time in Vienna. In 1897 he became a teacher in mathematics at the Upper High School of Trieste ("Civica Scuola Reale Superiore di Trieste"). In 1900 he was nominated professor for commercial and political arithmetic at the I.R. Accademia di Commercio e Nautica. He was the director of this institution from 1910 to 1937. Apparently, his reputation was overwhelming. In a to celebrate him, he was euphorically called "Eine Zierde der Menschheit!" and ""heroic scientist".[92]

7.4 *The Accademia in Trieste and the k.u.k. educational system*

The Accademia di Commercio e Nautica in Trieste was the oldest Accademia in the Habsburgian-Hungarin empire, and was founded in 1817 by the Austrian administration, in order to develop Trieste as the empire's seaport. Later, international insurance-companies like "Generali", Llyod Adriatico" and "Riunione Adriatica di Sicurta", corporations with a strong life-insurance-branch and with assets exceeding the assets of the big banking corporations of Vienna, recruited their employees from the Accademia. In this process, the Accademia also started to include courses on money, banking, finance, probability, etc. in the curriculum for students, and to offer evening courses for practitioners.

Trieste was the main port of the Austria-Hungarian empire at the Mediterranean, and thus the main trading place for commodities – both physically and in terms of derivatives. Even though trading declined during the last quarter of the 19th century, a majority of the citizens of Trieste still believed in the strength of their town as a commercial, social and cultural melting-point, and continued to attract an international clientele of businessmen, artists and scientists: Italians, Austrians, Slowens, Germans i.e. Business was key in amalgamating the different nations, and created a liberal, international atmosphere in the town. This sharply contrasted the emerging perception of "nationality". In his book *Il mio Carso,* the Triestinian writer Scipio Slatapter took the perspective of a man who wanted to maintain his Italian nationality:

> *"Ogni cosa al commercio necessaria è violazione d'italianità; ciò che ne è vero aumento danneggia quello."*[93] *(p. 181)*

This was the political atmosphere which prevailed at the time when Bronzin started his career in Trieste. The economic situation was still flourishing in the empire; the central government in Vienna, particularly the k.u.k. Administration, closely governed and controlled the publicly-owned institutions in the empire.

This was particularly true for the commercial education system. One of the driving forces behind this process was Eugenio Gelcich. He was the director of the Accademia in Trieste from 1899 to 1901, and thus, was responsible for the appointment of Bronzin in 1900. One year later, he became the central inspector of "commercial education" in the empire, which reflects the high standard and reputation the Accademia (and possibly other commercial schools) had in Trieste. He was a very involved person and visited by order of the Austrian government all European countries for studying their commercial education system and wrote reports about his visits.[94] He regularly organized conferences where the professors of all commercial academies and schools of the empire participated. In 1910 he even made Trieste to the permanent center of the interna-

[93] p 181; quoted after Ara/Magris (1987), p. 48.
[94] He wrote reports on the commercial education system in Italy, France, Greece, Switzerland, Hungaria, Belgium, Austria; see Subak (1917), p. 271.

tional association for developing the commercial education (see Subak 1917, pp. 269 ff.). In 1903, a curriculum for all advanced commercial schools (*höhere Handelsschulen*) of the empire was developed under his guidance. This constituted the basis for establishing a (the first?) standardized curriculum in business finance, including topics such as[96]

- Introduction to probability theory;
- Life annuities;
- Exchange traded commodities and securities;
- The system of exchanges and contracts/ instruments (commodities and securities; spot and forward; options).

This is a surprisingly modern course outline. Given Bronzin's strong educational efforts, we do not believe that he wanted to launch a new research program with his option pricing booklet, but rather provide a solid educational tool for accomplishing the standards defined above. It may be that this interpretation understates his real ambition, but the more amazing is what came out.

[96] *"Einführung in die Wahrscheinlichkeitsrechnung ... Leibrenten … Die börsenmässigen Handelsgeschäfte in Waren und Effekten … Einteilung der Börsen, die Geschäfte an Waren- und Effektenbörsen, Effektiv- und Termingeschäfte, … Prämiengeschäfte".* Source: Archivio dello stato di Trieste, atto *Accademia di commercio e nautica in Trieste*, b 101 e regg 273, cif A195, 1903.

8. *Summing up: Bronzin's contribution to option pricing*

When comparing Bronzin's contribution to Bachelier's thesis, then without any doubt, Bachelier was not only earlier, but his analysis is more rigorous from a mathematical point of view. Bronzin can not be credited having developed a new mathematical field, as Bachelier did with his theory on diffusions. Bronzin did *no* stochastic modeling, applied no stochastic calculus, derived no differential equations (except in the context of our equation 7), he was not interested in stochastic processes, and hence his notion of volatility has no time dimension. But except this, every element of modern option pricing is there! His contribution can be assessed as follows:

1. He noticed the unpredictability of speculative prices, and the need to use probability laws to price derivatives.
2. He recognized the informational role of market prices for pricing derivatives, and developed a theory relying on the current forward price of securities to price options. No expected values show up in the pricing formulas. His probability densities can be easily re-interpreted as risk-neutral pricing densities.
3. He understood the key role of arbitrage, although he is not very explicit about it; the derives the put-call parity condition, and uses a zero-profit condition to price forward contracts and options.
4. He develops a simplified procedure to find analytical solutions for option prices by exploiting a key relationship between their derivatives (with respect to their exercise prices) and the underlying pricing density. He also stresses the empirical advantages of this approach.
5. He extensively discusses how different distributional assumptions affect option prices. In particular, he shows how the normal law of error – which is the normal density function – can be

used to price options, and how it is related to a binomial stock price distribution.

6. Besides of pricing simple calls and puts, he develops formula for chooser options and, more important, repeat-options.

His preference-free valuation equation (43) is closer to the Black-Scholes formula than anything else published before Black, Scholes and Merton. All this is a remarkable achievement, and it is done with a minimum of analytics. There are few things on the less elegant side: the discussion and the large systems of hedging conditions in the first part belongs to it, and some numerical procedures to solve for the repeat-option premiums also. But nevertheless, Bronzin's contribution is important, not only in historical retro-perspective. He definitively deserves his place in the history of option pricing, as other researchers as well. This was pointed out in a survey article by Girlich (2002). His introduction concludes our paper:

> *"In the case of Louis Bachelier and his area of activity the dominant French point of view is the most natural thing in the world and every body is convinced by the results. The aim of the present paper is to add a few tesseras from other countries to the picture which is known about the birth of mathematical finance and its probabilistic environment."*

We are happy having added another piece to this fascinating picture. Bachelier in his way was unique. But what is more important: he was not the only one who was working successfully on option pricing at the beginning of the 20th century. So, the question remains why the results of Bachelier, Bronzin, and possibly other's yet to be rediscovered, did not get a broader acceptance? Why did their research not find immediate successors, academics that continued the way towards a practicable formula that made the pricing of options a subject of ongoing scientific research? Finding answers to these questions could help us to better understand the cultural background of financial mathematics, and would probably add an interesting chapter to the sociology of science. This would be a fascinating agenda of future research.

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