Simulation (part 2)

Computational Economics, spring term 2012

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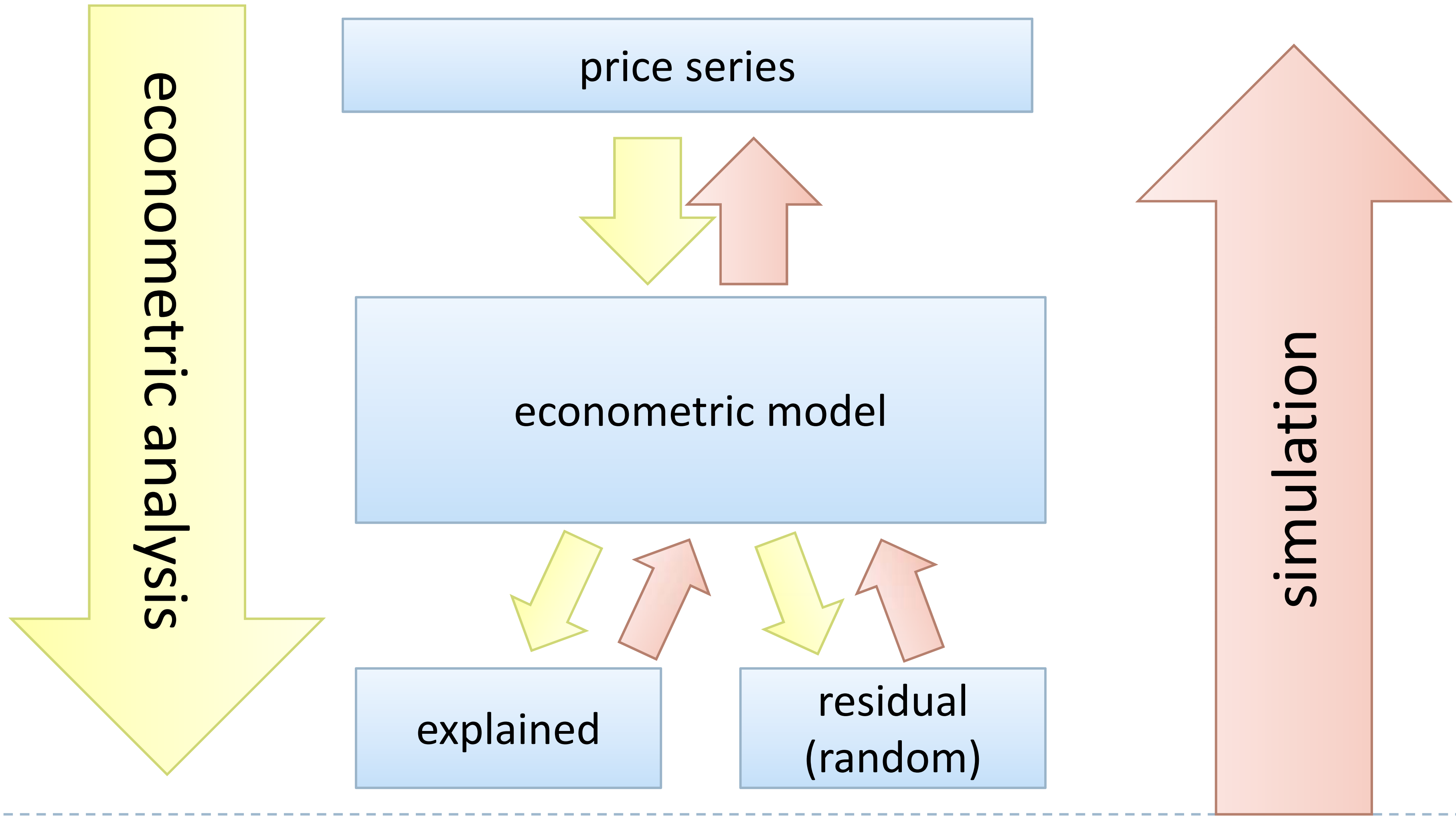
simulation with econometric models

- econometric modelling



simulation with econometric models

- assumptions about stock prices
  - past return: $r_t = \ln(S_t / S_{t-1})$
  - follow a stochastic process
    - $S_{t+1} = S_t \exp(r_{t+1})$
    - r_{t+1} is a „random variable"

- simplest case
  - normal distribution
    - mean and volatility are sufficient
    - return for 1 period: $r_{t+1} \sim N(\mu, \sigma)$
    - return for T periods: $r_{t+T} \sim N(\mu T, \sigma\sqrt{T})$
    - expected price: $E(S_{t+T}) = S_t \exp\left((\mu + \sigma^2/2)T\right)$

simulation with econometric models

econometric model

$$S_t = S_{t-1} \cdot e^{r_t}, r_t \sim \text{niid}(\mu, \sigma^2)$$

$$r_t = \mu + \varepsilon_t, \varepsilon_t \sim \text{niid}(0, \sigma^2)$$

$$\varepsilon_t = z_t \cdot \sigma, z_t \sim \text{niid}(0;1)$$

MC simulation

1. simulate $\tilde{z}_t$
2. compute $\tilde{\varepsilon}_t = \tilde{z}_t \cdot \sigma$
3. compute $\tilde{r}_t = \mu + \tilde{\varepsilon}_t$
4. compute $\tilde{S}_t = S_{t-1} \cdot e^{\tilde{r}_t}$

simulation with econometric models

- some simple return processes
  - returns follow a random walk:

$$r_t = \mu + \epsilon_t \qquad \epsilon_t \sim N(0, \sigma_\epsilon)$$

  - are driven by some factor:

$$r_t = \alpha + \beta r_{f,t} + \epsilon_t$$

  - have some memory (auto regressive):

$$r_t = \mu + \gamma r_{t-1} + \epsilon_t \qquad r_t = \mu + \theta_1 e_{t-1} + e_t$$

- time varying volatility
  - **G**eneralized **A**uto**R**egressive **H**eteroskedasticity (GARCH)

$$r_t = \mu + \varepsilon_t, \;\; \varepsilon_t \sim N\left(0; \sigma_t^2\right), \qquad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

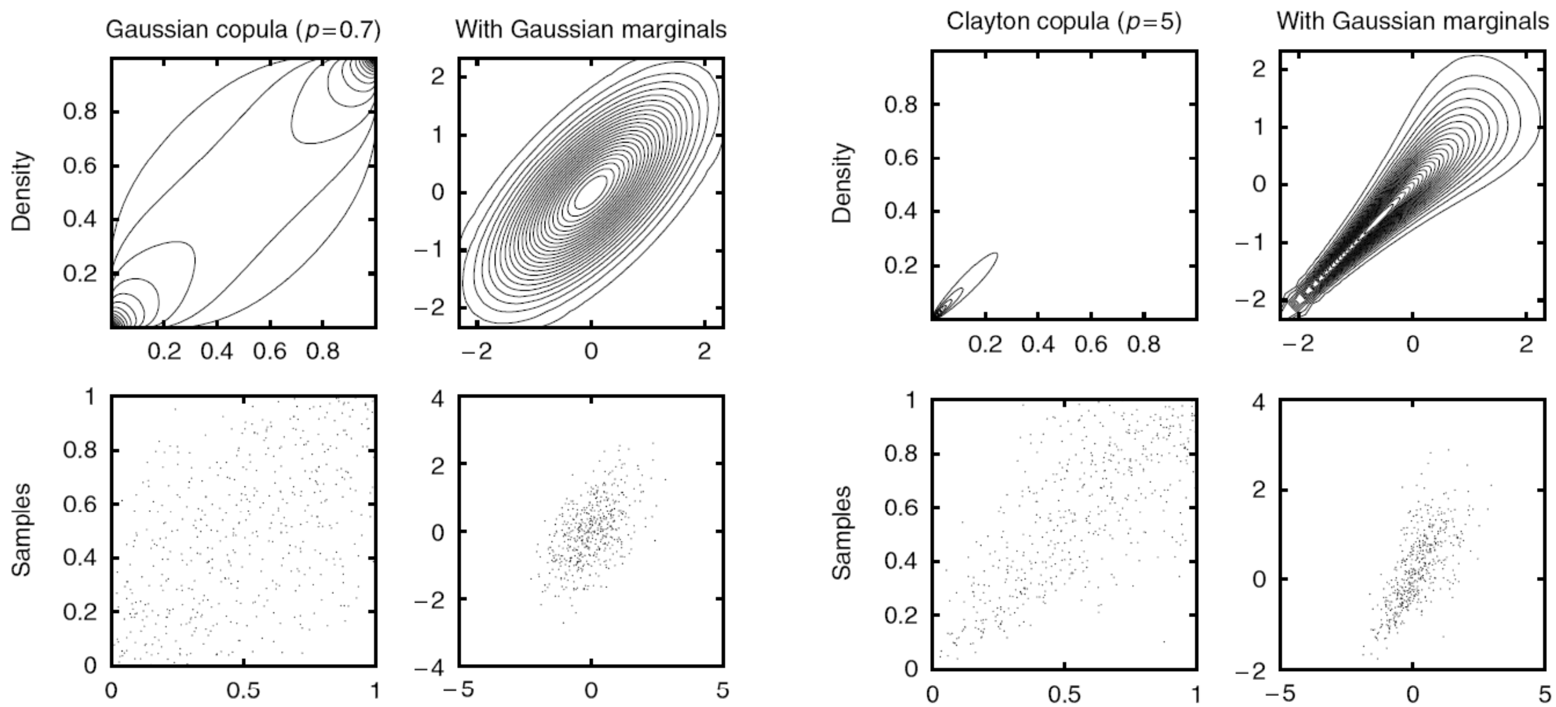
generating multivariate random numbers

- example: multivariate normal distribution
  - assume $\mathbf{X}$ is centered, i.e., $E(\mathbf{X}) = 0$, and normally distributed with $\text{cov}(\mathbf{X}) = \mathbf{C}$
  - let $\mathbf{C}^{-0.5} = (\mathbf{C}^{0.5})^{-1}$ and $(\mathbf{C}^{0.5})'\mathbf{C}^{0.5} = \mathbf{C}$
  - to get $\mathbf{C}^{0.5}$, use, e.g., the Cholesky decomposition
  - then $\mathbf{Z} = \mathbf{X}\,\mathbf{C}^{-0.5}$ is white noise with $\text{cov}(\mathbf{Z}) = \mathbf{I}$
  - likewise, $\text{cov}(\mathbf{Z}\,\mathbf{C}^{0.5}) = \text{cov}(\mathbf{X}\,\mathbf{C}^{-0.5}\,\mathbf{C}^{0.5}) = \text{cov}(\mathbf{X}) = \mathbf{C}$
  - $\text{cov}(\mathbf{X}\,\mathbf{C}^{-0.5}) = \mathbf{I}$ where $\mathbf{C}^{-0.5} = (\mathbf{C}^{0.5})^{-1}$ and $(\mathbf{C}^{0.5})'\mathbf{C}^{0.5} = \mathbf{C}$

- in Matlab
  - given: number of variables n, covariance matrix C (nxn) , number of samples S
  - `X = randn(S,n) * chol(C)`

generating multivariate random numbers

- example: copulas
  - background for arbitrary distribution, if $X \sim F$, then $F(X) \sim U(0,1)$
  - copula = joint density of uniform variables (i.e., of quantiles)
  - Sklar's theorem: $F(X,Y) = C(F(X), F(Y))$



generating multivariate random numbers

- example: arbitrary mv parametric distributions
  - individual samples: e.g., acceptance/rejection
  - sequences of samples: e.g., Metropolis(-Hastings), Gibbs-Sampler
- example: empirical distribution
  - e.g., semi-parametric: multivariate kernel density estimation
  - e.g., Taylor-Thompson:
    - randomly pick one observation and its (*m*-1) nearest neighbours $x_{j_1}, \ldots x_{j_m}$
    - new sample = linear combination of *m* points, randomly weighted

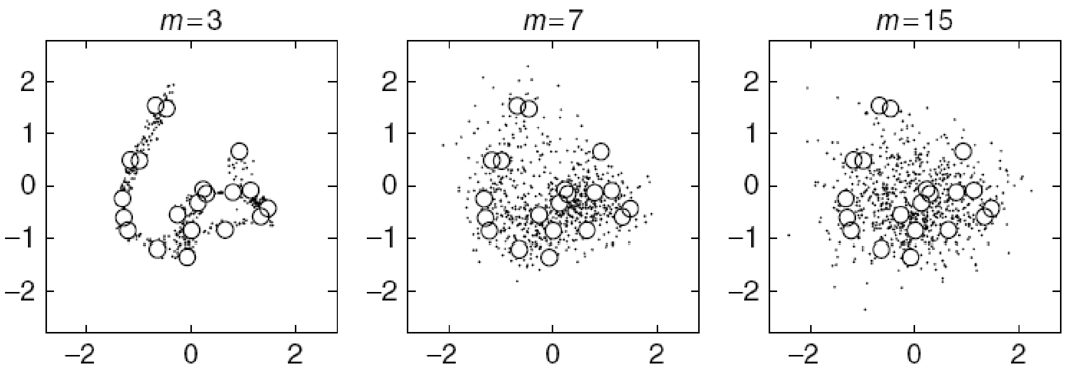


Figure 6.7 Samples generated with the Taylor–Thompson algorithm; original observations (circles) from a $(0,1)$ Gaussian distribution.

$$\bar{x}_s = \frac{1}{m} \sum_i x_{j_i}$$

$$u_i \sim U(-1,+1)$$

$$w_i = \frac{1 + \sqrt{3(m-1)}u_i}{m}$$

$$x_s = \bar{x}_s + \sum_i w_i (x_{j_i} - \bar{x}_s)$$

literature

- additional reading

  - J Gentle (2009), *Computational Statistics*, Springer.
  - Gilli, M., Maringer, D., and Schumann, E. (2011), *Numerical Methods and Optimization in Finance*, Academic Press.
  - O Jones, R Maillardet, A Robinson (2009), *Scientific Programming and Simulation Using R*, CRC Press.