3. Multistage Games

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1. Multistage Games with Observable Actions

1.1 Introduction

A game in extensive form models the dynamic structure of a strategic interaction by specifying

- the players.
- when each player has the move.
- what each player can do at each of his or her opportunities to move.
- what each player knows at each of his or her opportunities to move.
- the payoff received by each player for each combination of moves that could be chosen by the players.

The general model of a game in extensive form is quite complicated and we will thus restrict attention to the important special cases of multistage games with observable actions.
1. Multistage Games with Observable Actions

1.2 Building Blocks

- **Players:** $i = 1, \ldots, n$.
  - In most examples and applications we will consider 2-player games: $i = 1, 2$.

- **Stage $k = 1$:**
  - The game starts at stage 1 with the empty history $h^0 = \emptyset$.
  - The rules of the game specify a non-empty subset of players who are active at stage 1 and for each of these players a set $A^1_i$ of actions.
  - The active players simultaneously choose actions $a^1_i \in A^1_i$, resulting in an action profile $a^1$.
  - The history of the game after stage 1 is given by $h^1 = a^1$.
  - Depending on the history $h^1$ the game may either stop or continue to stage 2.
1. Multistage Games with Observable Actions

1.2 Building Blocks

- **Stage $k > 1$:**
  - Suppose the game has reached stage $k$ with a history $h^{k-1} = a^1, \ldots, a^{k-1}$.
  - The rules of the game specify a non-empty subset of players who are active at stage $k$ and for each of these players a set $A_i^k$ of actions. Both the set of active players and the sets of actions available to them may depend on the history $h^{k-1}$.
  - The active players simultaneously choose actions $a_i^k$, resulting in an action profile $a^k$ for the $k$-th stage of the game.
  - The history of the game after stage $k$ is given by $h^k = h^{k-1}, a^k$.
  - Depending on the history $h^k$ the game may either stop or continue to stage $k + 1$. 
1. Multistage Games with Observable Actions

1.2 Building Blocks

- Information: At every stage $k$ every active player knows the history $h^{k-1}$ of the game up to that stage.
  - This is the assumption of observable actions. The general model of a multistage game allows for the possibility that players have only partial information about the history of the game.

- Terminal Histories: A history is called terminal if the game stops after this history or the history has infinite lengths. Denote the set of all terminal histories by $\hat{H}$.
  - If all terminal histories have finite lengths, the game has a finite horizon. Otherwise the game has an infinite horizon.

- Payoffs for player $i = 1, \cdots, n$ are given by a payoff function $u_i : \hat{H} \to \mathbb{R}$, assigning a payoff to every terminal history.
1. Multistage Games with Observable Actions

1.3 Remarks

- The above building blocks describes a multistage game with observable actions and complete information.
  - To describe a multistage game with incomplete information add an initial stage at \( t = 0 \) in which nature draws types for the players as in a static game of incomplete information.
  - Until we get to the final part of the lecture we assume complete information.
- In many applications it is necessary to extend the above framework by introducing random moves.
  - For instance, a random move may determine which player is allowed to move first.
  - We will not consider such random moves.
1. Multistage Games with Observable Actions

1.4 Some Special Cases

- **Static games** with complete (resp. incomplete) information are a special case of a multistage game with complete (resp. incomplete) information in which
  - All players are active at stage 1.
  - The game stops after stage 1 no matter which action profile the players have chosen.

- **Multistage games with perfect information:** After every non-terminal history there is exactly *one* active player.
  - Hence, in a multistage game with perfect information players may move, say, one after the other or in an alternating order, but it is never the case that players move simultaneously.
  - In addition, whenever a player has the move, he knows all other actions that were chosen previously (observable actions).
  - It is often convenient to represent such games by a game tree.
1. Multistage Games with Observable Actions

1.5 The Strategic Form

- A strategy $s_i$ for a player in a multistage game assigns to every history after which the player is active an action which is feasible for the player after this history. Let $S_i$ denote the set of all strategies for player $i$.

- Every strategy profile $s = (s_1, \ldots, s_n) \in S$ determines a unique terminal history, called the outcome of the strategy profile and denoted by $h(s)$.

1. The strategy profile specifies an action for every player active at stage 1, resulting in the action profile $a^1$. If the game stops after $a^1$ the outcome has been determined. Otherwise the strategy profile causes the game to continue to stage 2 with history $h^1 = a^1$.

2. Suppose the strategy profile has caused the game to continue to stage $k > 1$ with history $h^{k-1}$. Given the history $h^{k-1}$, the strategy profile specifies an action for every player active at stage $k$, resulting in an action profile $a^k$. If the game stops after $h^k = h^{k-1}, a^k$ the outcome of the strategy profile has been determined. If the game does not stop after $h^k$ the strategy profile causes the game to continue to stage $k + 1$. 
1. Multistage Games with Observable Actions

1.5 The Strategic Form

- Define $U_i : S \rightarrow \mathbb{R}$ by $U_i(s) = u_i(h(s))$, i.e., the payoff player $i$ obtains from a strategy profile $s$ is the utility of the outcome $h(s)$ determined by $s$.

**Definition**

The game $G = (S_1, \ldots, S_n; U_1, \ldots, U_n)$ which is obtained from a given multistage game as described above is the **strategic form** of the multistage game.

**Note:**

- Using the strategic form, concepts such as strictly dominated strategies, best responses, and Nash equilibrium are well-defined for multistage games.
2. Multistage Games with Perfect Information

2.1. Example: Two Players, Two Stages

- Two players: $i = 1, 2$.
- At stage 1 player 1 moves and chooses an action $a_1 \in A_1$. No matter which action player 1 chooses, the game continues to stage 2.
- At stage 2 player 2 moves and chooses an action $a_2 \in A_2$. Thereafter the game stops.
  - Note that in this example the action set of player 2 does not depend on the previous choice of player 1.
- A terminal history is given by $(a_1, a_2) \in A_1 \times A_2$.
- Payoff functions are given by $u_i : A_1 \times A_2 \to \mathbb{R}$.
A strategy for player 1 is given by $a_1 \in A_1$.
A strategy for player 2 is given by a function $s_2 : A_1 \rightarrow A_2$ specifying an action $a_2 = s_2(a_1)$ for every possible choice of player 1 in the first stage.
The outcome of the strategy profile $(a_1, s_2)$ is given by $(a_1, s_2(a_1))$.
The payoff functions in the strategic form are given by
$U_i(a_1, s_2) = u_i(a_1, s_2(a_1))$.
2. Multistage Games with Perfect Information

2.2 Backwards Induction

Consider the example just introduced. The backwards-induction solution of such a game is the strategy profile \((a_1^*, R_2)\) determined as follows:

1. Determine for every \(a_1 \in A_1\) the action of player 2 which solves

\[
\max_{a_2 \in A_2} u_2(a_1, a_2)
\]

and call the solution of this problem \(R_2(a_1)\).

2. Determine the action of player 1 which solves

\[
\max_{a_1 \in A_1} u_1(a_1, R_2(a_1))
\]

and call the solution to this problem \(a_1^*\).

\((a_1^*, a_2^*) = (a_1^*, R_2(a_1^*))\) is the backwards-induction outcome of the game.
Remarks:

- The above definition assumes that both maximization problems considered have a unique solution.
- More generally, \((a_1^*, R_2)\) is a backwards-induction solution for the example if

\[
u_2(a_1, R_2(a_1)) \geq u_2(a_1, a_2) \text{ for all } (a_1, a_2)
\]

as well as

\[
u_1(a_1^*, R_2(a_1^*)) \geq u_1(a_1, R_2(a_1)) \text{ for all } a_1
\]

holds.
- If both \(A_1\) and \(A_2\) are finite, there exists at least one backwards-induction solution.
2. Multistage Games with Perfect Information

2.2 Backwards Induction

Backwards-induction in general multistage games with perfect information and a finite horizon:

1. Let \( m \) be the lengths of the longest history. Determine and record for all non-terminal histories \( h^{m-1} \) the optimal action of the player who has the move after history \( h^{m-1} \).

2. Eliminate all decisions at stage \( m \) from the game by assigning to non-terminal histories of length \( h^{m-1} \) the payoffs which result if the optimal actions determined in the previous step are taken in stage \( m \).

3. The result is a multistage game with perfect information and a finite horizon of length \( m - 1 \). Apply steps 1 and 2 to this game.

4. Repeat until optimal decisions are determined following all non-terminal histories.

The strategy profile determined by this procedure is the backwards-induction solution. The outcome of this strategy profile is the backwards-induction outcome.
Remarks:

- As in the example the definition can be generalized to deal with the possibility that optimal actions are not uniquely defined.
- If the strategic form of the multistage game is a finite game, a backwards-induction solution exists.
- Backwards-induction as defined above cannot be applied to multistage games with an infinite horizon.
Proposition

Suppose the strategy profile $s$ is a backwards-induction solution of a multistage game with perfect information and a finite horizon. Then $s$ is a Nash equilibrium of the strategic form of the multistage game.

Note:

- If a backwards-induction solution exists (as it does whenever the strategic form is a finite game) the corresponding strategic form has at least one Nash equilibrium in pure strategies.
- The strategic form may have Nash equilibria which do not correspond to a backwards-induction solution of the multistage game.
2. Multistage Games with Perfect Information
2.5 The Stackelberg Game

- Players: \( i = 1, 2 \).
- In stage 1 player 1 chooses \( q_1 \in \mathbb{R}_+ \).
- For all choices of \( q_1 \) the game continues to stage 2 and player 2 chooses \( q_2 \in \mathbb{R}_+ \). Thereafter the game stops.
- Payoffs are given by

\[
u_i(q_1, q_2) = [P(q_1 + q_2) - c] q_i,\]

where \( P(Q) = \max\{a - Q, 0\} \) with \( a > c \geq 0 \).

Note: Actions and payoff functions are the same as in the Cournot-Duopoly, but the timing differs.
2. Multistage Games with Perfect Information

2.5 The Stackelberg Game

Backwards-induction:

- Solve \( \max_{q_2 \geq 0} [P(q_1 + q_2) - c] q_2 \) to obtain

\[
R_2(q_1) = \begin{cases} 
0 & \text{if } q_1 \geq a - c, \\
\frac{a - c - q_1}{2} & \text{if } q_1 < a - c.
\end{cases}
\]

- This is identical to player’s 2 best response function from the Cournot-Duopoly.

- Solve

\[
\max_{q_1 \geq 0} [P(q_1 + R_2(q_1)) - c] q_1
\]

to obtain

\[
q_1^* = \frac{a - c}{2}.
\]

- \( (q_1^*, R_2) \) as determined above is the backwards-induction solution. The backwards induction outcome is given by

\[
q_1^*, R_2(q_1^*) = \frac{a - c}{2}, \frac{a - c}{4}.
\]
2. Multistage Games with Perfect Information

2.5 The Stackelberg Game

Note:

- The strategic form of this game has many Nash equilibria besides the backwards-induction solution.
- For instance, the following strategy profile is a Nash-equilibrium in the strategic form:

\[
q_1 = 0
\]

\[
s_2(q_1) = \begin{cases} 
\frac{a-c}{2} & \text{if } q_1 = 0, \\
(a-c) & \text{if } q_1 > 0.
\end{cases}
\]
3. Subgame Perfect Equilibrium

Reinhard Selten
Winner of the Nobel Prize in Economics 1994
Photo: Volker Lannert/University of Bonn

http://www.myscience.de/image/db/menu_17164.jpg
3. Subgame Perfect Equilibrium

3.1 Definition

- Every non-terminal history $h$ of a multistage game with observable actions defines a subgame, namely the multistage game with observable actions which is obtained by considering the situation after history $h$ as the first stage of a game.
  - Note that by this definition the original game is a subgame of itself. All other subgames are called proper subgames.
- Every strategy profile for the strategic form of the original game induces a strategy profile for the strategic form of every subgame.
  - To obtain the strategy profile for a subgame simply restrict the domain of the original strategy profile to those histories which are part of the subgame.

Definition (Subgame Perfect Equilibrium)

A strategy profile in a multistage game with observable actions is a **subgame perfect equilibrium** if it induces a Nash-equilibrium in every subgame of the original game.
3. Subgame Perfect Equilibrium

3.2 Remarks

- Subgame perfection is a refinement of Nash equilibrium: If a strategy profile is subgame perfect it is a Nash equilibrium, but the reverse need not hold.

- Subgame perfection is a generalization of backwards-indention: In multistage games with perfect information a strategy profile is subgame perfect if and only if it is a backwards induction solution.

- In a static game a strategy profile is subgame perfect if and only if it is a Nash equilibrium.
4. Two-Stage Games with Imperfect Information

4.1 Bank Runs

- Players: $i = 1, 2$.
- At stage 1 both players simultaneously decide whether ($a^1_i = 1$) or not ($a^1_i = 0$) to withdraw.
- After the history $a^1 = (0, 0)$ the game continues to stage 2. Otherwise it stops.
- If the game continues to stage 2 both investors again simultaneously decide whether ($a^2_i = 1$) or not ($a^2_i = 0$) to withdraw.
- After stage 2 the game stops.
4. Two-Stage Games with Imperfect Information

4.1 Bank Runs

- Payoff functions are given by

\[ u_1(1, 0) = u_2(0, 1) = D, \]
\[ u_1(0, 1) = u_2(1, 0) = 2r - D \]
\[ u_1(1, 1) = u_2(1, 1) = r \]

and

\[ u_1((0, 0), (1, 0)) = u_1((0, 0), (0, 1)) = 2R - D \]
\[ u_1((0, 0), (0, 1)) = u_1((0, 0), (1, 0)) = D \]
\[ u_1((0, 0), (1, 1)) = u_1((0, 0), (1, 1)) = R \]
\[ u_1((0, 0), (0, 0)) = u_1((0, 0), (0, 0)) = R, \]

where \( R > D > r > D/2 > 0 \).
4. Two-Stage Games with Imperfect Information

4.1 Bank Runs

- A strategy for player \( i \) can be described by \((a^1_i, a^2_i) \in \{0, 1\}^2\), where \( a^2_i \) is the action chosen by player 2 if the game continues to stage 2.

- The strategic form of the game is

<table>
<thead>
<tr>
<th></th>
<th>(0, 0)</th>
<th>(0, 1)</th>
<th>(1, 0)</th>
<th>(1, 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>( R, R )</td>
<td>( D, 2R - D )</td>
<td>( 2r - D, D )</td>
<td>( 2r - D, D )</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>( 2R - D, D )</td>
<td>( R, R )</td>
<td>( 2r - D, D )</td>
<td>( 2r - D, D )</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>( D, 2r - D )</td>
<td>( D, 2r - D )</td>
<td>( r, r )</td>
<td>( r, r )</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>( D, 2r - D )</td>
<td>( D, 2r - D )</td>
<td>( r, r )</td>
<td>( r, r )</td>
</tr>
</tbody>
</table>
The game has two subgames, namely the game itself and the proper subgame following the history $a^1 = (0, 0)$.

To identify subgame-perfect equilibria, begin by identifying Nash equilibria of all those subgames that start at stage 2. Here we only have to consider the subgame following the history $a^1 = (0, 0)$ with the strategic form:

$$
\begin{array}{c|cc}
0 & 0 & 1 \\
\hline
0 & R, R & D, 2R - D \\
1 & 2R - D, D & R, R \\
\end{array}
$$

Because $R > D$ and $2R - D > R$ the unique Nash equilibrium of this game is $a^2 = (1, 1)$. 
4. Two-Stage Games with Imperfect Information

4.1 Bank Runs

- After having identified Nash equilibria for the subgames starting at stage 2, remove all those strategies from the strategic form of the original game which do not specify equilibrium behavior at stage 2. Here the resulting game is:

<table>
<thead>
<tr>
<th></th>
<th>(0, 1)</th>
<th>(1, 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 1)</td>
<td>R, R</td>
<td>2r − D, D</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>D, 2r − D</td>
<td>r, r</td>
</tr>
</tbody>
</table>

- Because $R > D$ and $r > 2r − D$ this game has two Nash equilibria, namely $((0, 1), (0, 1))$ and $((1, 1), (1, 1))$.

- These are the two subgame perfect equilibria of the original game.
4. Two-Stage Games with Imperfect Information

4.2 Market Entry

- Players $i = 1, 2$.
- At stage 1 both players simultaneously decide whether ($a^1_i = 1$) or not ($a^1_i = 0$) to enter a market.
- After the history $a^1 = (0, 0)$ the game stops.
- After the history $a^1 = (1, 0)$ the only active player at stage 2 is player 1. He chooses $q_1 \in \mathbb{R}_+$. 
- After the history $a^1 = (0, 1)$ the only active player at stage 2 is player 2. He chooses $q_2 \in \mathbb{R}_+$. 
- After the history $a^1 = (1, 1)$ both players are active at stage 2 and simultaneously choose quantities $q_i \in \mathbb{R}_+$. 
- After stage 2 the game stops.
4. Two-Stage Games with Imperfect Information
4.2 Market Entry

- In any terminal history in which a player chooses $a_i^1 = 0$ he obtains a payoff of zero.
- Otherwise the payoff of player $i$ is given by

$$
\pi_i(q_1, q_2) = P(q_1 + q_2)q_i - F,
$$

where $P(Q) = \max\{1 - Q, 0\}$ with $F > 0$ and $q_j = 0$ holds if $a_j^1 = 0$. 

4. Two-Stage Games with Imperfect Information

4.2 Market Entry

A strategy for player $i$ is given by $(a_i^1, q_i^m, q_i^c)$ where $a_i^1$ is his choice at stage 1, $q_i^m$ is his choice at stage 2 if he is the only player to have chosen 1 at stage 1, and $q_i^c$ is his choice after the history $a^1 = (1,1)$.

This game has three proper subgames at stage 2:

1. The subgame starting with the history $a^1 = (1,1)$. Here $(q_1^c, q_2^c) = (\frac{1}{3}, \frac{1}{3})$ is the unique Nash equilibrium.

2. The subgame starting with the history $a^1 = (1,0)$. Here $q_1^m = \frac{1}{2}$ is the unique Nash equilibrium.

3. The subgame starting with the history $a^1 = (0,1)$. Here $q_2^m = \frac{1}{2}$ is the unique Nash equilibrium.
4. Two-Stage Games with Imperfect Information
4.2 Market Entry

- To determine the subgame perfect equilibria, it remains to determine for which combinations \((a_1^1, a_2^1)\) the strategy combination

\[
(a_1^1, 1/2, 1/3), (a_2^1, 1/2, 1/3)
\]

is a Nash equilibrium in the game in strategic form which results if all strategies that are not of the above form are eliminated.

- The resulting game can be written as:

\[
\begin{array}{c|cc}
 & 0 & 1 \\
\hline
0 & (0, 0) & (0, 1/4 - F) \\
1 & (1/4 - F, 0) & (1/9 - F, 1/9 - F) \\
\end{array}
\]
4. Two-Stage Games with Imperfect Information

4.2 Market Entry

- If $F > 1/4$ the unique Nash equilibrium of this game is $(0, 0)$, so that

$$((0,1/2,1/3), (0,1/2,1/3))$$

is the unique subgame perfect equilibrium of the two-stage game.

- If $F < 1/9$ the unique Nash equilibrium of the above game is $(1, 1)$, so that

$$((1,1/2,1/3), (1,1/2,1/3))$$

is the unique subgame perfect equilibrium of the two-stage game.

- If $1/4 > F > 1/9$ the strategy profiles $(1, 0)$ and $(0, 1)$ are Nash equilibria in the above game, so that

$$((1,1/2,1/3), (0,1/2,1/3)) \text{ and } ((0,1/2,1/3), (1,1/2,1/3))$$

are subgame perfect equilibria of the two-stage game.