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Sorting in Iterated Incumbency Contests*

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Abstract: This paper analyzes iterated incumbency contests with heterogeneous valuations in a large population setting. Incumbents repeatedly face different challengers, holding on to their positions until defeated in a contest. Defeated incumbents turn into challengers until they win a contest against an incumbent, thereby regaining an incumbency position. We consider steady-state equilibria of this process and study how and to which extend individuals sort into the incumbency positions depending on their valuations. In particular, we identify sufficient conditions for positive sorting, meaning that the share of individuals with a given valuation holding an incumbency position is increasing in the valuation, and provide an example to show that negative rather than positive sorting may arise in equilibrium. Further results show how incumbency rents and sorting are affected by the frequency at which incumbency is contested and the scarcity of the incumbency positions.

Keywords: Contests, Sorting, Incumbency Rents, Steady-State Equilibrium.

***JEL* Classification Numbers:** C72, D72, D74.

1 Introduction

Repeated incumbency contests describe situations in which an incumbent, who enjoys an incumbency rent, faces a sequence of challenger each of whom seeks to displace the current incumbent in a contest. Such repeated incumbency contests occur, for example, when incumbent politicians have to secure reelection after every legislative period or when a ruler has to repeatedly defend his territory against external aggressors. Other forms of repeated incumbency contests involve CEOs that have to uphold their positions in the face of recurrent competition from rivals, or firms that have to stand their ground in market niches against potential entrants. Sports champions can reap benefits while being the leader but have to constantly fight against challengers to stay on top.

As is suggested by these examples, repeated incumbency contests come in various forms: In some, the incumbent once defeated by a challenger leaves the game forever, as is the case e.g. with medieval kings but also with modern despots generally not surviving successful uprisings. This is the situation analyzed in Stephan and Ursprung (1998) and Virág (2009). On the other hand, there are situations in which the defeated

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incumbent becomes the challenger in the following period. Such is the situation analyzed in Mehlum and Moene (2006) who consider repeated incumbency contests between two agents that switch positions whenever the incumbent looses. The political competition in two-party systems is an example of this form of repeated incumbency contests.

There are many instances of repeated incumbency contests, however, where the defeated incumbent neither leaves the game forever nor returns to challenge the very position that he has just been evicted from. An example illustrating this is the market for CEOs, who usually stay in the market after losing the job to a competitor and sooner or later appear again as CEO in a different firm. Alternatively, one may think of the incumbents as entrepreneurs whose firms occupy market niches which accommodate at most one firm. Once an entrepreneur has successfully entered a niche she has to defend it against the intrusion of rival firms, and here, too, the presumption is that an entrepreneur whose firm is driven out of its niche proceeds to searching for alternative ways of doing business. The model that we consider in this paper captures this turnover in the incumbency positions by supposing that there are many incumbency positions and defeated incumbents return to the pool of the challengers that search for other incumbency positions to contest. We refer to this kind of scenario as an iterated incumbency contest.

We assume that the valuations for the incumbency positions are heterogeneous. One would then expect that individuals with higher valuations are inclined to fight harder in any given contest and, consequently, will have a higher likelihood to win contests and, therefore, a higher likelihood of being an incumbent rather than a challenger. The main purpose of this paper is to test the intuition that such positive sorting into the incumbency positions must arise in equilibrium. More generally, we are interested in the factors determining the extent of sorting in equilibrium and in studying how the iterated nature of the contests affects (relative) incumbency rents.[1]

In our model there is a continuum of identical incumbency positions, with these positions distributed among members of a unit mass population. Each individual can hold at most one incumbency position at a given time. We refer to individuals that do so as incumbents, and individuals that do not as challengers. Time is continuous. Individuals meet other individuals, randomly sampled from the population, at a fixed, exogenous rate. Whenever a meeting is between an incumbent and a challenger, a contest ensues. If the current incumbent wins the contest, then the challenger awaits her next opportunity to contest an incumbent. If the current incumbent loses the contest, then the challenger takes over the position and the incumbent joins the pools of challengers. We study steady states in which the strategies and the utilities of the players are time-invariant.[2]

[1]Related questions are analysed in Virág (2009) who considers a model in which there is only one incumbency position and, as already mentioned above, considers the case in which defeated incumbents leave the game forever. Contests are modelled as an all-pay auction with private values, with values drawn from a continuous distribution. The focus in Virág (2009) is on the question whether sorting must be perfect in a stationary equilibrium in the sense that the incumbent distribution is concentrated on the highest possible type. His main finding is that such perfect sorting does not obtain even in the limit when the arrival rate of challengers becomes infinite. In contrast, our analysis focuses on situations for which it is transparent that perfect sorting cannot arise in equilibrium.

[2]Our focus on steady states differentiates our analysis from the one in Hafer (2006), who studies the transitional dynamics from an arbitrary initial situation. Further important difference to Hafer (2006) are that (i) in her setting investments into contest efforts cease after the transitional period, whereas in our setting costly contests are a recurring phenomenon in steady-state, (ii) she, like Virág (2009), considers a setting with incomplete information, whereas we assume complete information, and (iii) she restricts attention to the war of attrition when modeling the interaction between challenger and

Individuals come in a finite number of types, differing in their commonly known flow payoff obtained when being incumbent. The masses of these types, that we refer to as the population distribution, are exogenous. Given the probability that an individual of a given type retains or acquires the incumbency position in a contest with an individual of some other type, the population distribution determines, as we show, the incumbent distribution, that is, the masses of the different types that hold an incumbency position. These probabilities of winning the contest are in turn determined by the incumbency rents. Crucially, these incumbency rents are endogenous, as they not only depend on the flow payoff obtained while being incumbent, but also on the rents that are lost in defending the incumbency position as well as the rents that are gained by attacking an incumbency positions – with these later two terms depending on the incumbency rents of other agents and the incumbent distribution.

As regards the contests, we do not restrict attention to a particular contest model but capture the structure of a variety of contests by specifying the contest payoffs and the winning probability of the challenger directly as functions of the incumbency rents of the two contestants. We impose a number of regularity conditions on these functions: (i) both the winning probabilities and the payoffs are continuous in the rents, (ii) payoffs are consistent with the assumptions that either contestant can forego the incumbency position at zero cost and that neither contestant receives more than her rent in case of winning, and (iii) the contestant's winning probabilities are strictly increasing in own rents. This approach is convenient as we can abstract from the details of the contests in our general analysis, but still relate to particular contest models when discussing our assumptions and results. Specifically, our assumptions on the contest outcomes encompass the standard complete information contests, namely the Tullock contest (Tullock, 1980) and the all-pay auction (Baye et al., 1996). We discuss other contests from the literature that fit our framework, including sequential move and multiple-stage contests, when introducing the formal model in Section 2.1. For the time being, we refer the reader to Konrad (2009) who provides an exhaustive survey of contests with complete information.

Equilibrium is characterized by an incumbent distribution specifying a time-invariant mass of incumbents for each of the different types and a incumbency rent profile specifying a time-invariant incumbency rent for each of the different types. Equilibrium requires that the incumbent distribution and the rent profile correspond to a steady state and are consistent with optimal behavior in the contests. Our first main result (Proposition 1) establishes that equilibrium exists.

We then turn to the question of sorting in equilibrium. Proposition 2 establishes sufficient conditions on the structure of the contests which ensure that positive sorting obtains. These conditions entail that the challenger's expected gain from engaging in the contest is increasing in her rent and, similarly, that the incumbent's expected loss from engaging in the contest is increasing in her rent. These monotonicity conditions on expected gains and losses ensure that higher types have higher equilibrium rents. Together with the assumption that the winning probabilities are strictly increasing in the own rent, which ensures that types sort into the incumbency conditions according to their rents, these conditions ensure positive sorting. The monotonicity conditions of Proposition 2 are satisfied in the standard cases of the all-pay auction as well as of the Tullock contest.

Following Virág (2009), we also consider the limit when the meeting process be-

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incumbent, whereas our main results impose relatively little structure on the incumbency contests.

comes frictionless, so that both incumbents and challengers are continually engaged in contests. With such continual contests the incumbency rents of all types vanish under fairly general conditions (Lemma 5). Somewhat surprisingly, for a broad class of contests, including the all-pay auction and the Tullock contest, strictly positive sorting nevertheless obtains in the limit (Proposition 3) and the surplus generated by the availability of incumbency positions is not fully dissipated (Proposition 4).

For the special case of an iterated all-pay auction with two types we establish uniqueness of the equilibrium (Proposition 5) and discuss the comparative statics with respect to the frequency of meetings and the abundance of the incumbency positions. In particular we can show that sorting becomes more pronounced when we increase the meeting rate in settings where the incumbency positions are scarce, yet that it becomes less pronounced when the incumbency positions are abundant (Proposition 6). Furthermore, adding a fixed cost of attack to the all-pay auction allows us to construct an example which satisfies the monotonicity condition on the gain of the challenger in Proposition 2 but violates that on the incumbent loss and features both an equilibrium with positive sorting and an equilibrium with negative sorting (Proposition 7). In the later equilibrium individuals with lower flow payoff from the incumbency position obtain higher incumbency rents because they face a lower probability of being dislodged from their position. We view this example as important as it shows that the sufficient conditions of Proposition 2 indeed have bite.

From a modelling perspective, the most novel aspect of this paper is to use a search model with a large population to model iterated contests. Embedding pairwise interactions into a dynamic framework via the use of a search-model is, of course, a well-established modelling strategy in other contexts. For surveys of the vast literature concerned with search models of the labor market we refer the reader to Pissarides (1990) or Rogerson et al. (2005). Closer to our concerns is the literature on sorting in search-and-matching models with heterogenous agents, recently surveyed in Chade et al. (forthcoming). Relative to this second strand of literature, the most important distinguishing feature of our analysis arises from the very nature of incumbency contests, namely that for incumbents meetings do not generate but destroy rents and that sorting does not arise from agent's decisions which matches to accept but from the intensity with which agents fight to maintain or change their positions.[3] Finally, we note that researchers in behavioral ecology have considered steady-state models of territorial conflicts (Grafen, 1987; Eshel and Sansone, 1995, 2001; Kokko et al., 2006) that bear some similarity to our work but differ in that the model of the conflict arising when an incumbent and a challenger meet is very rudimentary (namely the Hawk-Dove game in which the contestants simply decide whether to fight or not) and that there are no counterparts to our positive sorting results.

The remainder of the paper is organized as follows: Section 2 describes our model of iterated contests and defines equilibrium. Section 3 establishes existence of equilibrium , derives sufficient conditions for positive sorting, and analyzes the case of continual contests. Section 4 offers a detailed analysis of the iterated all-pay auction with two-types and shows, in particular, how extending the all-pay auction by adding an an attack cost for the challenger may give rise to equilibria with negative sorting. Section 5 concludes. All proofs and some supplementary materials are in the appendix.

[3]The feature that meetings may destroy rents is also present in the otherwise rather different model of marriage and divorce investigated in Cornelius (2003) in which agents may unilaterally decide to leave a current relationship upon encountering a more attractive partner, thereby leaving their current partner worse off.

2 The Model

Our model has two main building blocks: a description of equilibrium behavior in a given incumbency contest as a function of the contestants' values of winning or losing the contest and a population framework with heterogenous agents in which incumbency contests arise over time from random meetings between incumbents and challengers. Section 2.1 introduces the first of these building blocks, Section 2.2 the second. Section 2.3 then ties the building blocks together by defining equilibrium. In a nutshell, equilibrium requires that the distributions of agents who hold incumbency positions is in steady state and that all agents' steady-state continuation values of winning or losing an incumbency contest are consistent with equilibrium behavior in the subsequent contests they will face.

2.1 Incumbency Contests

We consider two-player contests between an incumbent I and a challenger C. A contest is parameterized by specifying both contestants' values for winning or losing the contest. We denote these values by $(W_I, L_I) \in \mathbb{R}^2_+$ for the incumbent and by $(W_C, L_C) \in \mathbb{R}^2_+$ for the challenger and assume that winning is more attractive than losing: $W_I > L_I$ and $W_C > L_C$. The parameters are assumed to be common knowledge among the contestants. Both contestants are risk neutral. If the challenger wins the contest with probability p and loses it with probability $1-p$, then the challenger's payoff is $pW_C - (1-p)L_C - \kappa_C$ and the incumbent's payoff is $(1-p)W_I + pL_I - \kappa_I$, where κ_C and κ_I are the costs the two contestants incur in their fight for the incumbency position.

Let

$$x_I = W_I - L_I > 0 \text{ and } x_C = W_C - L_C > 0 \tag{1}$$

denote the player's incumbency rents. We model the outcome of the contest as a function of these rents, thereby gaining the flexibility to accommodate a variety of different contests in our formal analysis. In particular, we suppose that any contest between an incumbent with values (W_I, L_I) and a challenger with values (W_C, L_C) has a unique equilibrium with expected payoff $L_C + \sigma(x_I, x_C)$ and winning probability $\mu(x_I, x_C)$ for the challenger and expected payoff $W_I - \tau(x_I, x_C)$ and winning probability $1 - \mu(x_I, x_C)$ for the incumbent.[4] We are thus expressing equilibrium payoffs in terms of the challenger's expected gain $\sigma(x_I, x_C)$ from participating in the contest (compared to the benchmark of simply obtaining L_C) and the incumbents' expected loss $\tau(x_I, x_C)$ from participating in the contest (compared to the benchmark of simply obtaining W_I).[5]

Throughout our analysis we impose the following three assumptions on the expected gains, expected losses, and winning probabilities.

Assumption 1. *The functions $\sigma : \mathbb{R}^2_{++} \to \mathbb{R}$, $\tau : \mathbb{R}^2_{++} \to \mathbb{R}$, and $\mu : \mathbb{R}^2_{++} \to [0, 1]$ are continuous.*

Assumption 2. *The expected gains $\sigma(x_I, x_C)$ and the expected losses $\tau(x_I, x_C)$ satisfy*

$$0 \leq \sigma(x_I, x_C) \leq x_C, \quad 0 \leq \tau(x_I, x_C) \leq x_I \tag{2}$$

[4]The presumption that equilibrium in any given contest is unique is restrictive. We impose it to focus on the effects that arise from challengers and incumbents being repeatedly engaged in such contests.

[5]While unconventional, expressing equilibrium payoffs in terms of expected gains for challengers and expected losses for incumbents will greatly simplify the exposition once we embed incumbency contests into our population framework.

and

$$\sigma(x_I, x_C) - \tau(x_I, x_C) \leq \max\{0, x_C - x_I\} \tag{3}$$

for all $(x_I, x_C) \in \mathbb{R}^2_{++}$.

Assumption 3. *The challenger's winning probability* $\mu(x_I, x_C)$ *is strictly decreasing in* x_I *and strictly increasing in* x_C.

Assumption 1 serves to ensure the existence of equilibria in our population model (cf. Proposition 1 in Section 3.1). It is also required for the limit analysis we conduct in Section 3.3.

To interpret Assumption 2, we begin by observing that the inequality $0 \leq \sigma(x_I, x_C)$ states that the challenger is assured to obtain at least her value from losing the contest as her equilibrium payoff. This condition is satisfied in all standard contest models (e.g., because the challenger can refuse to enter the contest or can choose zero effort at zero cost). At the other extreme the inequality $\sigma(x_I, x_C) \leq x_C$ indicates that the challenger can never expect to gain more than her incumbency rent from participating in the contest. The interpretation of the inequalities $0 \leq \tau(x_I, x_C) \leq x_I$ appearing in (2) is analogous: On the one hand, the incumbent cannot lose more than her incumbency rent from participating in the contest (e.g., because she can walk away from the incumbency position when threatened by an attack). On the other hand, it is not possible for her to obtain an equilibrium payoff that is higher than her value of winning the contest. Upon adding $L_I + W_C$ on both sides of it, inequality (3) is easily seen to be equivalent to the statement that the sum of the two contestants equilibrium payoffs is smaller than $\max\{L_I + W_C, L_C + W_I\}$, which is a natural consequence of the fact that there is only one incumbency position to be filled, so that in any outcome of the contest there is one winner and one loser. If the two contestants have identical rents $x_I = x_C = y$, then inequality (3) reduces to $\sigma(y, y) - \tau(y, y) \leq 0$, with strict inequality indicating that there is some rent-dissipation in such a symmetric contest. In case both $\sigma(y, y) = 0$ and $\tau(y, y) = y$ holds, we have full rent-dissipation when the contestants have identical rents.

Recalling that the incumbent's winning probability is $1 - \mu(x_I, x_C)$, Assumption 3 is equivalent to the requirement that for both contestants their equilibrium probability of winning the contest is strictly increasing in their own incumbency rents. This is a substantive assumption in that it excludes the possibility that a contestant's winning probability is constant in her own rent over some range - as will be the case, for instance, if the equilibrium of the contest involves a boundary solution in which the challenger has a zero probability of winning the contest when x_C is sufficiently small. We return to this point in Section 4.4, where we explain why such violations of Assumption 3 sever the link between individual contests and the population structure which is at the heart of our paper.

Before we proceed, we give two examples of complete-information contests that fit our assumptions, namely the familiar all-pay auction and the equally familiar Tullock contest. These contests will serve as leading examples throughout our subsequent analysis.

Contest 1 (All-Pay Auction). *In the standard all-pay auction with complete information, I and C simultaneously sink efforts* $e_I, e_C \geq 0$ *at a cost equal to the chosen efforts. A player wins the contest for sure if he chooses the higher effort, and ties are broken by a fair coin toss.*

It is well-known (Baye et al., 1996) that this contest has a unique equilibrium, that is in mixed strategies. In this equilibrium, player C wins the contest with probability

$$\mu(x_I, x_C) = \begin{cases} \frac{1}{2}\frac{x_C}{x_I} & \text{if } x_C < x_I \\ 1 - \frac{1}{2}\frac{x_I}{x_C} & \text{if } x_C \geq x_I \end{cases}. \tag{4}$$

and the equilibrium payoffs are $\max\{x_I - x_C, 0\} + L_I$ *for player I and* $\max\{x_C - x_I, 0\} + L_C$ *for player C. In terms of the functions* σ *and* τ *we thus have*

$$\sigma(x_I, x_C) = \max\{x_C - x_I, 0\} \tag{5}$$

$$\tau(x_I, x_C) = x_I - \max\{x_I - x_C, 0\}. \tag{6}$$

Contest 2 (Tullock Contest)**.** *In a Tullock contest with complete information, I and C simultaneously sink efforts* $e_I, e_C \geq 0$ *at a cost equal to the chosen efforts. Given such effort choices player C wins with probability*

$$p(e_I, e_C) = \begin{cases} \frac{e_C^r}{e_I^r + e_C^r} & \text{if } e_I + e_C > 0 \\ \frac{1}{2} & \text{if } e_I + e_C = 0 \end{cases}, \tag{7}$$

and player I wins with probability $1 - p(e_I, e_C)$*.*

Under the parameter restriction $0 < r \leq 1$ *a unique pure strategy equilibrium exists,*[6] *in which (Nti, 1999)*

$$\sigma(x_I, x_C) = \frac{x_C^{r+1}}{\left(x_I^r + x_C^r\right)^2}\left[x_C^r + (1-r)x_I^r\right] \tag{8}$$

$$\tau(x_I, x_C) = x_I - \frac{x_I^{r+1}}{\left(x_I^r + x_C^r\right)^2}\left[x_I^r + (1-r)x_C^r\right] \tag{9}$$

and

$$\mu(x_I, x_C) = \frac{x_C^r}{x_I^r + x_C^r}. \tag{10}$$

Throughout the following, when we refer to the Tullock contest, we mean a Tullock contest satisfying the parameter restriction $0 < r \leq 1$*.*

It is easily verified that for the all-pay auction and for the Tullock contest Assumptions 1 - 3 are satisfied. Further, these two contests are homogeneous and role symmetric in the sense of the following definitions. These two properties play an important role in our subsequent analysis.

Definition 1. *(Homogeneity) A contest is homogenous if* $\sigma(x_I, x_C)$ *and* $\tau(x_I, x_C)$ *are both homogenous of degree* 1 *in* (x_I, x_C) *and* $\mu(x_I, x_C)$ *is homogenous of degree* 0 *in* (x_I, x_C)*.*

[6] More generally, a unique pure strategy equilibrium exists when $x_I^r + x_C^r > r\min\{x_I^r, x_C^r\}$, but as x_C and x_I are endogenous in our subsequent analysis, we work with the assumption $0 < r \leq 1$, which ensures this inequality for all strictly positive x_I and x_C.

Definition 2. *(Role Symmetry) A contest is role symmetric if*

$$\mu(y,z) = 1 - \mu(z,y) \tag{11}$$

and

$$\sigma(y,z) + \tau(z,y) = z \tag{12}$$

hold for all $(y,z) \in \mathbb{R}^2_{++}$.

Condition (11) in the definition of role-symmetry states that a contestant's winning probability does not depend on her role in the contest (i.e., whether she is the incumbent or the challenger) but is solely determined by her own incumbency rent and the incumbency rent of her opponent. The interpretation of (12) is less obvious but makes an analogous statement for contest payoffs. Specifically, consider an agent who has value W for winning and value L for losing the contest and who faces an opponent with incumbency rent y. If the agent under consideration is the incumbent, then her contest payoff is given by $W - \tau(W-L, y)$. If she is the challenger, then her contest payoff is $L + \sigma(y, W-L)$. Equating these expressions yields $\sigma(y, W-L) + \tau(W-L, y) = W - L$, which, upon denoting the agent's incumbency rent $W - L$ by z, is equation (12).

Examples of contests complying with Assumptions 1 - 3 that are homogeneous but not role-symmetric include both all-pay auctions and Tullock contests with asymmetric bid-effectiveness for the challenger and the incumbent (Franke et al., 2014a; Leininger, 1993). Examples of contests that satisfy neither homogeneity nor role symmetry include contests with asymmetric head-starts as analyzed by Konrad (2002) and Siegel (2014) in an all-pay auction framework and by Franke et al. (2014b) in a Tullock contest framework. Alas, these contests violate our Assumption 3: if one of the players enjoys a head start that is sufficiently high, then both players remain inactive, implying that, for both players, the winning probabilities stay constant in their respective incumbency rents over a range.

Our formulation of a contest is not confined to simultaneous move one-shot contests, but also covers contests complying with Assumption 1 - 3 where players move sequentially (Baik and Shogren, 1992; Leininger, 1993) or contest where players simultaneously sink efforts in multiple periods along the lines of Yildirim (2005). As is the case for contests with head-starts (and for essentially the same reason), contests where the players first have to sink a fixed cost in order to participate as in Fu et al. (2015) violate Assumption 3 when such participation costs are deterministic. In Section 4.4 we provide an example of a contest featuring stochastic entry costs that satisfies Assumptions 1 – 3 and is neither homogenous nor role-symmetric.

2.2 Population Framework

There is a unit mass of risk-neutral and infinitely lived individuals. Time is continuous. All individuals discount future payoffs at rate $\rho > 0$. At each moment in time an individual either holds an incumbency position or not and will be referred to as an incumbent or a challenger accordingly. The mass of incumbency positions is fixed and given by $\theta \in (0,1)$.

Individuals come in $n \geq 2$ different, exogenous types labeled by $i \in N = \{1, \ldots, n\}$. The type of an individual determines the flow payoff $v_i > 0$ received while holding an incumbency position. Types are ordered such that $v_1 < v_2 < \ldots < v_n$. When not holding an incumbency position all individuals receive a flow payoff of zero. As we explain in more detail below, individuals transit from being incumbents to being challengers and vice versa by engaging in contests.

The proportion of type i individuals is given by $f_i > 0$. We refer to the vector $f = (f_i, ..., f_n)$, which satisfies $\sum_i^n f_i = 1$, as the population distribution. To simplify the exposition, we impose the genericity condition

$$\sum_{j \in J} f_j \neq \theta \text{ for all } J \subset N. \tag{13}$$

We will consider steady states in which the masses of incumbents and challengers of the different types are time-invariant and all incumbency positions are held by some agent, so that at each moment in time a fraction θ of the individuals are incumbents whereas the remaining fraction $1 - \theta$ of the individuals are challengers. We use $g_i \geq 0$ to denote the mass of incumbents of type i in such a steady-state, whereas $f_i - g_i \geq 0$ is the corresponding mass of challengers. Let

$$\mathcal{G} = \{g \in \mathbb{R}^n_+ : g_i \leq f_i, \forall i \in N \text{ and } \sum_{i=1}^{n} g_i = \theta\}. \tag{14}$$

We refer to a vector $g = (g_1, \ldots, g_n) \in \mathcal{G}$ as an incumbent distribution.

Incumbency contest arise when challengers meet incumbents. For simplicity, we suppose that meetings between individuals are random and generated by a quadratic search technology (Diamond and Maskin, 1979) with exogenous meeting rate $m > 0$: Thus, each challenger meets incumbents of type $i \in N$ at rate mg_i and each incumbent meets challengers of type $i \in N$ at rate $m(f_i - g_i)$. Every meeting between an incumbent and a challenger triggers an incumbency contest. If the incumbent wins the contest, both contestants retain their current roles. If the challenger wins the contest, she obtains the incumbency position, whereas the previous incumbent becomes a challenger.[7] For any agent, the value of of winning a contest is thus given by the continuation value of being an incumbent and the value of losing a contest is given by the continuation value of being a challenger. These continuation values must not only take into account the flow payoffs (v_i for an incumbent of type i and 0 for a challenger) of being in the particular role but also the expected losses (for an incumbent) and expected gains (for a challenger) arising from future contests.

2.3 Equilibrium

As mentioned previously, we restrict attention to steady states. Further, we only consider steady states which are type-symmetric in the sense that any two individuals with the same type have identical continuation payoffs when being in the same role. In line with the notation introduced in Section 2.1, we denote the continuation payoff of an incumbent of type i by W_i and the continuation payoff of a challenger of type i by L_i and let $x_i = W_i - L_i$ denote the incumbency rent of an individual of type i, which we take to be strictly positive.

[7] Assuming that every meeting between an incumbent and a challenger triggers a contest for the incumbency position is without loss of generality provided that both agents in the meeting have strictly positive incumbency rents. Condition (2) in Assumption 2 then ensure that both agents are at least as well off from participating in the contest rather than avoiding it (say, by the challenger choosing not to attack or the incumbent choosing not to defend her position). The proviso that incumbency rents are strictly positive will be part of our equilibrium definition. See also Remark 1 at the end of Section 2.3.

Throughout the following we take for granted that payoffs and winning probabilities in every contest are determined as discussed in Section 2.1. In particular, if a challenger of type i encounters an incumbent of type j, then the resulting gain for the challenger is $\sigma(x_j, x_i)$, the resulting loss for the incumbent is $\tau(x_j, x_i)$, and the winning probability for the challenger is $\mu(x_j, x_i)$. In defining equilibrium we can thus focus on the determination of the incumbent distribution $g \in G$ and an incumbency rent profile $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$ specifying an incumbency rent for each type. Together the incumbent distribution and the incumbency rent profile provide sufficient information to determine all the continuation values (cf. equations (16) and (17) below).

Our first equilibrium requirement is that the incumbency distribution is in steady-state. The rates at which individuals of different types switch from being challengers to being incumbents and vice versa are jointly determined by the incumbent distribution g and the vector of incumbency rents x, with the former determining the meeting rates and the later determining the winning probability of a challenger. In particular, the rate at which challengers of type i obtain incumbency positions is given by $\sum_j g_j \mu(x_j, x_i)$ and the rate at which incumbents of type i lose incumbency positions (and thereby turn into challengers) is given by $\sum_{j \in N}(f_j - g_j)\mu(x_i, x_j)$. To maintain a steady state, the inflow and the outflow to and from incumbency must balance for each type. Hence, steady state requires the following balance conditions:

$$(f_i - g_i) \sum_{j \in N} g_j \mu(x_j, x_i) = g_i \sum_{j \in N} (f_j - g_j) \mu(x_i, x_j), \quad \forall i \in N. \tag{15}$$

Our second equilibrium requirement is that the expected gains and losses individuals obtain from future interactions generate continuation values which are consistent with the incumbency rents presumed in calculating expected gains and losses. Given an incumbency rent profile x and an incumbent distribution g, the continuation value L_i of a challenger of type i is given by

$$L_i = \frac{m}{\rho} \sum_{j \in N} \sigma(x_j, x_i) g_j. \tag{16}$$

This is so because the flow payoff of a challenger is zero, implying that the continuation value of a challenger is nothing but the expected value of the gains arising from the possibility of contesting an incumbency position. As meetings with an incumbent of type j lead to an expected payoff of $L_i + \sigma(x_j, x_i)$, the expected gain of such a meeting is $\sigma(x_j, x_i)$. Hence, taking all potential meetings and the rates mg_j at which they occur into account, the expected gain is given by the right side of (16). (Observe that by Assumption 2 the expression $\sigma(x_j, x_i)$ is positive for all i and j, ensuring that it is indeed optimal - as we have presumed before - for a challenger to engage in any opportunity to contest an incumbency position.) Similarly, taking the flow payoff v_i and the expected value of the losses arising from being drawn into a contest into account, the continuation value of an incumbent of type i is given by

$$W_i = \frac{v_i}{\rho} - \frac{m}{\rho} \sum_{j \in N} \tau(x_i, x_j)(f_j - g_j). \tag{17}$$

Equations (16) and (17) embody the requirement that continuation values are determined by the flow payoffs and expected gains and losses accruing to challengers and incumbents. To turn this into an equilibrium condition in terms of an incumbency rent profile and incumbent distribution, we recall the identity $x_i = W_i - L_i$ and subtract (16) from (17) to obtain the value equations

$$\rho x_i = v_i - m \sum_{j \in N} \tau(x_i, x_j)(f_j - g_j) - m \sum_{j \in N} \sigma(x_j, x_i) g_j, \quad \forall i \in N. \tag{18}$$

Together with the the balance conditions (15), the value equations (18) define equilibrium (with the associated equilibrium continuation values given by (16) and (17)):

Definition 3 (Equilibrium)**.** *An equilibrium is a tuple* $(x, g) = (x_1, \ldots, x_n, g_1, \ldots, g_n) \in \mathbb{R}^n_{++} \times \mathcal{G}$ *satisfying the balance conditions* (15) *and the value equations* (18)*.*

Remark 1 (Strict Positivity of Rents)**.** *We have made it part of our equilibrium definition that all incumbency rents are strictly positive and have used this requirement to argue that it is always optimal for both challengers and incumbents to engage in a contest upon meeting each other (cf. Footnote 7). The existence result in Proposition 1 below provides justification for this approach. Nevertheless, one may wonder what happens if we had taken the possibility of negative (including zero) incumbency rents into account. Let us thus suppose that there is some type i for which the continuation value of being a challenger weakly exceeds the continuation value of being an incumbent. When in the role of a challenger an individual of type i would then weakly prefer not to engage in any contests and, presuming that doing so is possible, thus obtain a continuation value of zero. On the other hand, presuming that incumbents can also avoid participating in any contests by abandoning the incumbency position upon meeting a challenger, the continuation payoff of any incumbent of type i must be strictly positive as such an incumbent can simply collect the strictly positive flow payoff* v_i *until first meeting a challenger. Consequently, the same considerations that motivate Assumption 2 in Section 2.1 preclude the possibility of negative incumbency rents arising in our population framework. Observe, though, that zero incumbency rents will appear in the limit of continual contests that we consider in Section 3.3.*

3 Sorting in Equilibrium

3.1 Preliminary Results

Before turning to the substance of our analysis, we make two observations. The first observation is that equilibrium exists:

Proposition 1. *An equilibrium exists.*

The proof of Proposition 1 is (as are all other proofs for Section3) in Appendix A. It consists in a fairly standard application of Brouwers' fixed-point theorem, using the continuity of the functions σ, τ and μ on their domains to obtain the requisite continuity of the fixed-point map. The one difficulty in the proof is that the domain of these functions excludes the boundary of $\mathbb{R}^2_+$. As in the related existence proof for search-and-matching models with a finite number of types in Lauermann and Nöldeke (2015), this difficulty can be dealt with by using appropriate boundary conditions, which are here provided by Assumption 2.

In general, there is no assurance that equilibrium is unique. Indeed, Section 4.4 presents an example with multiple equilibria and the only result establishing uniqueness of equilibrium that we have been able to obtain is for the case of the all-pay auction (cf.

Proposition 5 in Section 4.2). What we can say, however, is that for a given incumbency rent profile there is a unique incumbent distribution solving the balance conditions (15), providing us with a counterpart to the fundamental matching lemma from Shimer and Smith (2000). For later reference we state this observation as a lemma.

Lemma 1. *For every $x \in \mathbb{R}^n_{++}$, there is a unique $g \in \mathcal{G}$ such that (x, g) solves the balance conditions* (15) *and this g is strictly positive.*

The proof of Lemma 1 adapts arguments that Banaji and Baigent (2008) have developed to establish uniqueness of an equilibrium in a model of an electron transfer networks to our model. Besides the continuity of the functions μ, the key property required for the argument is an implication of Assumption 3, namely that the challenger's winning probability is strictly positive no matter what the types of the challenger and the incumbent are. It is this implication which also yields that any g solving the balance conditions is strictly positive, that is, satisfies $g_i > 0$ for all $i \in N$.

3.2 Sufficient Conditions for Positive Sorting

We are interested in determining whether equilibrium induces positive sorting in the sense that the share of individuals of a given type who hold incumbency positions is strictly increasing in type. This is captured in the following definition:

Definition 4 (Positive Sorting)**.** *There is positive sorting in equilibrium (x, g) if g_i/f_i is strictly increasing in i.*

Recalling that individuals with higher types obtain higher flow payoffs while holding the incumbency position, there is a two-part intuition for why one would expect positive sorting in equilibrium. The first part is that one would expect individuals with higher flow payoffs to be more eager to obtain the incumbency position, that is, to have higher incumbency rents. The second part is that (due to Assumption 3) contestants with higher incumbency rents are more likely to win any incumbency contest they engage in and should thus be overrepresented among incumbents.

We begin our analysis by verifying that the second part of this intuition is correct.

Lemma 2. *In any equilibrium (x, g) there is sorting by rents: $x_i > x_j$ implies $g_i/f_i > g_j/f_j$.*

We now turn to the first part of the above intuition, namely that one would expect higher incumbency flow payoffs to imply higher incumbency rents. This intuition is always correct when the threat of being challenged pales into insignificance for the incumbents, that is, the meeting rate is sufficiently small. In that case it follows from the value equations (18) that incumbency rents are approximately proportional to flow payoffs and therefore increasing in type. More formally, we obtain:

Lemma 3. *Let $m \in (0, \underline{m})$, where $\underline{m} = \rho \min_{i=1,\ldots n-1} \left[\frac{v_{i+1}}{v_i} - 1\right] > 0$. Then x_i is strictly increasing in i in any equilibrium (x, g).*

When the iterated nature of the contests has a significant impact on incumbency rents, the link between the flow payoffs v_i and the incumbency gains x_i is more subtle. However, a positive link between the flow payoffs v_i and the incumbency gains x_i is ensured when both the expected gain of a challenger and the expected loss of an incumbent are increasing in own rents:

Lemma 4. *If $\sigma(x_I, x_C)$ is increasing in x_C and $\tau(x_I, x_C)$ is increasing in x_I, then the incumbency rents x_i are strictly increasing in i in any equilibrium (x, g).*

Requiring that the challenger's expected gain $\sigma(x_I, x_C)$ from contesting an incumbent is increasing in the challenger's incumbency rent does not seem like much of an assumption. Indeed, we are not aware of any contest model in which this property does not hold. The requirement that the incumbent's expected loss $\tau(x_I, x_C)$ from an engaging in a contest is increasing in the incumbent's rent is less obvious. In particular, this requirement will be violated if a higher rent for the incumbent has a sufficiently strong discouragement effect on possible challengers. We will explore this issue in more detail in Section 4.4, where we demonstrate by example that the conclusion of Lemma 4 does not obtain if the condition on τ is dropped. On the other hand, for role-symmetric contests, the identity (12) implies that $\tau(x_I, x_C)$ is increasing in x_I if $\sigma(x_I, x_C)$ is increasing in x_C at a rate bounded above by one. Therefore, for role-symmetric contests Lemma 4 is applicable whenever an increase in a contestant's incumbency rent is partly dissipated by incurring higher costs in the the contest so that the contestant's expected gain increases by less than the incumbency rent.

Combining Lemmas 2, 3, and 4, the following sufficient conditions for positive sorting in equilibrium are immediate.

Proposition 2. *There is positive sorting in any equilibrium (x, g) if (i) $0 < m < \underline{m}$ holds or (ii) $\sigma(x_I, x_C)$ is increasing in x_C and $\tau(x_I, x_C)$ is increasing in x_I.*

Inspection of (5) and (6) shows that for the all-pay auction the sufficient condition (ii) for positive sorting in Proposition 2 is satisfied. As demonstrated in the proof of the following corollary, the same is true for the Tullock contest.

Corollary 1. *Suppose the incumbency contest is an all-pay auction or a Tullock contest. Then the incumbency rents x_i are strictly increasing in i in any equilibrium (x, g) and there is positive sorting in any equilibrium.*

3.3 Continual Contests

Lemma 3 in Section 3.2 has considered the case in which contests for the incumbency position occur only rarely because the meeting rate m is low. We have seen that for such sporadic contests positive sorting always obtains in equilibrium (Proposition 2). In particular, in the limit $m \to 0$ incumbency rents converge to $x_i = v_i/\rho$ and the equilibrium sorting pattern is determined by the solution - which we have shown to be unique in Lemma 1 - to the balance conditions (15) given these incumbency rents. Here we consider the opposite extreme of continual contests, i.e., the limit as $m \to \infty$. Formally, we study limit equilibria in the sense of the following definition.

Definition 5 (Limit Equilibrium)**.** *A tuple $(x, g) \in \mathbb{R}^n_+ \times \mathcal{G}$ is a limit equilibrium if there exists a sequence $(m^k, x^k, g^k)_{k=1}^{\infty} \in \mathbb{R}_{++} \times \mathbb{R}^n_{++} \times \mathcal{G}$ such that for all k the tuple (x^k, g^k) is an equilibrium given the meeting rate m^k, the sequence of meeting rates converges to infinity, and the sequence $(x^k, g^k)_{k=1}^{\infty}$ converges to (x, g).*

Note that the definition of a limit equilibrium does not require the limit incumbency rents to be strictly positive. Indeed, our first result is that quite generally incumbency rents vanish as the meeting rate goes to infinity:

Lemma 5. *Suppose that (i) $\sigma(y, y) > 0$ holds for all $y > 0$ or (ii) the contest is role-symmetric. Then every limit equilibrium (x, g) satisfies $x = 0$.*

The conclusion obtained in Lemma 5 is intuitive: ultimately, incumbency rents stem from the ability to enjoy the flow payoff associated with holding the incumbency position. As the meeting rate goes to infinity incumbency becomes constantly contested, thereby eliminating the advantage of holding the incumbency position. Alas, this conclusion might fail if there were some type who (for finite meeting rates) never has a chance of obtaining a strictly positive gain when in the role of a challenger and, simultaneously is effectively insulated from any competitive pressure when holding the incumbency position. Condition (i) in the statement of Lemma 5 eliminates this possibility by ensuring that a challenger obtains a strictly positive gain when contesting an incumbent with the same, strictly positive incumbency gain. As indicated by condition (ii) this requirement can be dispensed with when the contest is role-symmetric. The reason is that if $\sigma(y, y) = 0$ holds in a role symmetric contest (as would be the case for all $y > 0$ in an all-pay auction, cf. equation (5)), then it is the case that there is full rent-dissipation if two contestants with incumbency rents y are matched with each other.[8] In conjunction with the genericity requirement (13) this suffices to imply the desired result.

The result that incumbency rents vanish with continual contests, suggests two conjectures. The first is that even if positive sorting holds for all finite meeting rates, it vanishes in the limit as m converges to infinity. That is, the ratios g_i/f_i all converge to θ. The second is that the aggregate flow payoff $\sum_{i \in N} g_i v_i$ accruing to the holders of the incumbency positions is fully dissipated in the same limit. That is, the aggregate surplus

$$S = \sum_{i \in N} [(f_i - g_i)L_i + g_i W_i] = \sum_{i \in N} [f_i L_i + g_i x_i] \,, \tag{19}$$

converges to zero. In the following we show that under fairly mild additional assumptions both of these conjectures are false for homogenous contests.

Proposition 3. *Suppose the contest is homogenous with $\sigma(x_I, x_C)$ increasing in x_C and $\tau(x_I, x_C)$ increasing in x_I and that condition (i) or (ii) from Lemma 5 holds. Then there is positive sorting in every limit equilibrium (x, g).*

As the proof of Proposition 3 makes clear, the reason why positive sorting persists in the limit is that even though all incumbency rents converge to zero as the meeting rate goes to infinity, they do so at the same rate and in such a way that the ratio of the incumbency rents x_j/x_i remains strictly increasing in j. As for homogeneous contests the ratios x_j/x_i uniquely determine the sorting pattern arising from the balance conditions, this suffices to deliver the result. We view homogeneity of the contest as the most substantive of the assumptions appearing in the statement of the proposition. The role of the additional conditions is simply to ensure that there is positive sorting along the equilibrium sequence converging to the limit equilibrium and that the arguments proving Lemma 5 are applicable to make the appropriate inferences for the convergence of the incumbency rents.

To state our next result about surplus dissipation for continual contests, it will be convenient to say that there is incomplete surplus dissipation in a limit equilibrium (x, g) if for any sequence of equilibria $(x^k, g^k)_{k=1}^{\infty}$ converging to (x, g) in the sense of Definition 5 every limit point of the associated sequence of equilibrium surpluses $(S^k)_{k=1}^{\infty}$ is strictly

[8] To see this, observe that from (12) we have $\sigma(y, y) + \tau(y, y) = y$ in every role-symmetric contest. Therefore $\sigma(y, y) = 0$ implies $\tau(y, y) = y$, so that the challenger's equilibrium payoff is L_C and the incumbent's equilibrium payoff is L_I.

positive. Formally, this associated sequence of equilibrium surpluses is defined by

$$S^k = \sum_{i \in N} \left[f_i \frac{m^k}{\rho} \sum_{j \in N} \sigma(x_j^k, x_i^k) g_j^k + g_i^k x_i^k \right], \tag{20}$$

where we have substituted (16) into the rightmost expression from (19) to obtain an expression depending only on (m^k, x^k, g^k). With this terminology in place, we can state the following result:

Proposition 4. *Suppose the conditions from Proposition 3 are satisfied and $\sigma(x_I, x_C) > 0$ holds for $x_I < x_C$. Then there is incomplete surplus dissipation in every limit equilibrium (x, g).*

The proof of Proposition 4 proceeds by showing that while the incumbency rents x_i^k and also the expected gains $\sigma(x_j^k, x_i^k)$ in equation (20) all converge to zero, the additional condition $\sigma(x_I, x_C) > 0$ for $x_I < x_C$ in the statement of the proposition ensures that challengers with the highest type have strictly positive continuation payoffs in the limit. The intuition for this result is that incumbency rents for the highest type disappear not because incumbents of this type face continual contests but rather because with frequent meetings challengers of the highest type find it very easy to obtain incumbency positions.[9]

Both the all-pay auction and the Tullock contest are role symmetric and, as we have noted before (cf. the discussion preceding the statement of Corollary 1) satisfy the monotonicity conditions on σ and τ appearing in the statement of Proposition 3. Further, it is immediate from (5) and (8) that the additional condition appearing in Proposition 4 holds for both of these contests, too. Hence, we may state without further proof:

Corollary 2. *Suppose the contest is an all-pay auction or a Tullock contest. Then there is positive sorting and incomplete surplus dissipation in every limit equilibrium (x, g).*

4 The All-Pay Auction with Two Types

In this section we conduct a more detailed study of the equilibria in the all-pay auction with two types. Focusing on this special case allows us provide insights into the structure of equilibria of iterated contests that go beyond the sufficient conditions for positive sorting and the limit results presented in the previous section. In particular, we can establish uniqueness of equilibrium and and derive the comparative statics with respect to the meeting rate m. In addition, the all-pay auction with two-types provides a convenient building block for the construction of an iterated contest which allows us to illustrate the possibility of equilibrium multiplicity and negative sorting. All proofs for this section are in Appendix B.

[9] Observe that this conclusion hinges on the fact that, as we have assumed, there are at least two different types of individuals. The all-pay auction without heterogeneity of types provides a simple example of a contest satisfying all the conditions of Proposition 4 but leading to complete surplus dissipation in every limit equilibrium.

4.1 Equilibrium Conditions

From Proposition 1, Lemma 1, and Corollary 1 we know that an equilibrium $(x, g))$ for the all-pay auction with two-types exists and every equilibrium satisfies

$$x_2 > x_1 > 0, \quad g_1 > 0, \quad g_2 > 0. \tag{21}$$

Using the winning probabilities for the all-pay auction (4) and the inequality $x_2 > x_1$, the balance conditions (15) reduce to[10]

$$g_2(f_1 - g_1) = \left[2\frac{x_2}{x_1} - 1\right] g_1(f_2 - g_2). \tag{22}$$

Similarly, substituting the expected gains (5) and losses (6) for the all-pay auction into the value equations (18) and using $x_2 > x_1$, the value equations reduce to

$$\rho x_1 = v_1 - m(1-\theta)x_1 \tag{23}$$
$$\rho x_2 = v_2 - m(1-\theta)x_2 - m(2g_1 - f_1)(x_2 - x_1), \tag{24}$$

where we have used $f_1 + f_2 = 1$ and

$$g_1 + g_2 = \theta \tag{25}$$

to simplify the resulting expressions. An equilibrium of the all-pay auction with two types is thus given by a tuple (x_1, x_2, g_1, g_2) satisfying the inequalities in (21) and solving the equations (22) - (25).

4.2 Uniqueness of Equilibrium

For the all-pay auction the expected gain for a challenger with the lowest incumbency rent x_1 is always zero and the expected loss for such an incumbent is always equal to the incumbency rent. Therefore, as indicated by equation (23), the incumbency rent for type 1 is simply the present value $v_1/(\rho + m(1-\theta)$ of the flow payoff v_1, where the discount rate is incremented by $m(1-\theta))$ to take the complete loss of the incumbency rent into account that results whenever an incumbent of type 1 meets a challenger. In particular, the equilibrium value of x_1 is independent of x_2, so that for given g_1 the value equations (23) - (24) can be solved recursively beginning with x_1 and then moving on to x_2.[11] Substituting the resulting value for the ratio x_2/x_1 into (22) and eliminating g_2 from this equation by using (25) then yields one equation in g_1. As we show in Appendix B the resulting equation has a unique solution in the interval $[\max\{\theta - f_2, 0\}, \min\{f_1, \theta\}]$. This suffices to imply uniqueness of equilibrium as, first, (x_1, x_2) and g_2 are uniquely determined by g_1 and, second, for g_1 outside of the indicated interval either the condition $g_1 > 0$ or the condition $g_2 > 0$ from (21) is violated.

Proposition 5. *Let the contest be the all-pay auction and let $n = 2$. Then there exists a unique equilibrium (x, g).*

[10] Observe that with only two types it is always the case that the balance conditions for both types are equivalent to $g_2(f_1 - g_1)\mu(x_2, x_1) = g_1(f_2 - g_2)\mu(x_1, x_2)$. Inserting the winning probabilities for the all-pay auction yields (22).

[11] This insight extends to the all-pay auction with an arbitrary number of types: given that incumbency rents are increasing in type, the value equations can be solved recursively to determine the unique vector x of incumbency rents that are consistent with a given incumbent distribution g.

4.3 Comparative Statics

In this section we investigate the question whether a higher frequency of meetings promotes or impedes positive sorting. To do so, we measure the extent of sorting by the ratio g_2/g_1. For given θ, f_1, and f_2 this seems a natural measure of sorting in the two-type model we consider here, because an increase in this ratio indicates that the fraction of type 2 individuals who hold an incumbency position has increased, whereas the fraction of type 1 individuals who hold an incumbency position has decreased.

For the all-pay auction with two types, it is immediate from (22) that for given θ, f_1, and f_2 the extent of sorting occurring in equilibrium is determined by the ratio x_2/x_1: higher values of the ratio x_2/x_1 cause an increase in the right side of (22). As the sum of g_1 and g_2 equals θ, restoring the balance condition then requires a decrease in g_1 and an increase in g_2, so that the extent of sorting increases. Identifying conditions under which the extent of sorting is increasing (resp. decreasing) in the meeting rate m is thus tantamount to identifying conditions under which the ratio x_2/x_1 is increasing (resp. decreasing) in m in the (by Proposition 5) unique associated equilibrium.

Considering the value equations (23) and (24) it is evident that the equilibrium ratio x_2/x_1 is independent of m and equal to v_2/v_1 if $2g_1 - f_1 = 0$ holds. Substituting v_2/v_1 into (22) we obtain that in equilibrium x_2/x_1 is independent of m and equal to v_2/v_1 if

$$\theta = \theta^* \equiv \frac{1}{2} + \frac{1}{2}\left(1 - \frac{v_1}{v_2}\right)(1 - f_1) \in (0, 1). \tag{26}$$

As one would expect (and the proof of the following proposition verifies), when incumbency positions are relatively scare ($\theta < \theta^*$ holds), then the equilibrium incumbent distribution features a lower mass of type 1 agents than when $\theta = \theta^*$ holds. This implies $2g_1 - f_1 < 0$, which in turn implies that the equilibrium ratio x_2/x_1 is not only strictly larger than the ratio of the flow payoffs v_1/v_2 but also (as we show) strictly increasing in m. As a consequence, we obtain the result that for relatively scare incumbency positions more frequent conflicts lead to more pronounced sorting. Vice versa, the extent of sorting decreases in m when incumbency positions are relatively abundant:

Proposition 6. *Let the contest be the all-pay auction, let $n = 2$, and let θ^* be as given in* (26)*.*

(i) If $\theta < \theta^*$, then the ratio of rents x_2/x_1 and the extent of sorting in equilibrium is strictly increasing in m.

(ii) If $\theta = \theta^*$, then the ratio of rents x_2/x_1 and the extent of sorting in equilibrium is constant in m.

(iii) if $\theta > \theta^*$, then the ratio of rents x_2/x_1 and the extent of sorting in equilibrium is strictly decreasing in m.

Proposition 6 shows that the extent of sorting is not simply driven by the differences in flow payoffs, but depends in a subtle way on both the meeting rates and the availability of incumbency positions which shape the competitive environment in which the individual contests are embedded.

4.4 Negative Sorting and Multiple Equilibria in the All-Pay Auction with Attack Costs

In this section we consider the following extension of the all-pay auction.

Contest 3 (All-Pay Auction with Stochastic Attack Costs)**.** *The contest is given by the following game: Whenever a challenger and an incumbent meet, the challenger first draws a cost $c \geq 0$ from a probability distribution $H(c)$ which is continuous on its support $[0, \bar{c}]$ and has a mass point at 0, that is $H(0) > 0$. The realization of the cost level is observed by both contestants. The challenger then decides whether to attack the incumbent or not. If the challenger decides not to attack, then no further interaction takes place, neither player incurs any cost and both individuals retain their current roles. If the challenger attacks, then the all-pay auction as described in Section 2.1 is played with the challenger incurring the additional cost c.*

Imposing the refinement that the challenger attacks in case of indifference, this game has a unique subgame-perfect equilibrium with

$$\sigma(x_I, x_C) = \begin{cases} 0 & \text{if } x_I > x_C \\ H(x_C - x_I) \cdot [x_C - x_I - E[c|c \leq x_C - x_I]] & \text{if } x_I \leq x_C \end{cases} \quad (27)$$

$$\tau(x_I, x_C) = \begin{cases} H(0) \cdot x_C & \text{if } x_I > x_C \\ H(x_C - x_I) \cdot x_I & \text{if } x_I \leq x_C \end{cases} \quad (28)$$

$$\mu(x_I, x_C) = \begin{cases} H(0) \cdot \dfrac{1}{2} \dfrac{x_C}{x_I} & \text{if } x_I > x_C \\ H(x_C - x_I) \cdot \left[1 - \dfrac{1}{2} \dfrac{x_I}{x_C}\right] & \text{if } x_I \leq x_C \end{cases} \quad (29)$$

The functions $\mu(x_I, x_C)$, $\sigma(x_I, x_c)$, and $\tau(x_I, x_c)$ associated with this contest satisfy Assumptions 1 - 3. Indeed, this is the point of modelling attack costs as being drawn from a distribution $H(c)$ satisfying the stated conditions, with the continuity of the attack-cost distribution ensuring Assumption 1 and the mass point at $c = 0$ ensuring that the winning probability $\mu(x_I, x_C)$ is strictly positive for all $x_C > 0$, which is a prerequisite for Assumption 3 to hold.

Because Assumptions 1 - 3 hold, existence of equilibrium for the all-pay auction with stochastic attack costs is assured (Proposition 1) and every equilibrium features sorting by rents (Lemma 2). Therefore, every equilibrium of such a contest in which higher types obtain strictly higher incumbency rents must feature positive sorting. For meeting rates larger than the bound $\underline{m}$ given in Lemma 3 we are, however, not assured that this must be the case. The reason is that Lemma 4 is not applicable: While the expected gain in (27) is increasing in x_C, the expected loss in (28) fails to be increasing in x_I. To see the latter claim, fix x_C satisfying the inequality $x_C > \bar{c}/(1 - H(0))$ and consider that an incumbent with $x_I = x_C - \bar{c}$ will always be attacked by the challenger with rent x_C and thus faces an expected loss of $\tau(x_I, x_C) = x_C - \bar{c}$. On the other hand, an incumbent with $x'_I = x_C > x_I$ is only attacked by the challenger with rent x_C if the cost realization is $c = 0$ and therefore has an expected loss of $\tau(x'_I, x_C) = H(0)x_C$. From the inequality $x_C > \bar{c}/(1 - H(0))$ we thus have $\tau(x'_I, x_C) < \tau(x_I, x_C)$ for $x'_I > x_I$. Intuitively, what happens is that $\tau(x_I, x_C)$ fails to be increasing in x_I because stronger incumbents benefit from a discouragement effect, which reduces the probability that a challenger with a given rent x_C will attack them.

It is thus the case that Proposition 2 does not exclude the existence of an equilibrium in which the incumbency rent of the low type exceeds the incumbency rent of the high type (recall that we consider $n = 2$), that is, $x_1 > x_2$ holds. In such a constellation, Lemma 2 then implies that such an equilibrium features negative rather than positive sorting, that is, $g_1/f_1 > g_2/f_2$ holds.

The failure of the monotonicity requirement on $\tau(x_I, x_C)$ is indeed enough to invalidate the conclusion of Proposition 2. Specifically, it is possible to construct an example of an all-pay auctions with stochastic attack costs featuring an equilibrium with negative sorting. This equilibrium exists along with an equilibrium that entails positive sorting, so that the example also demonstrates how the iterated nature of the contest may give rise to multiple equilibria with strikingly different properties. To construct such an example, we suppose that the masses of the two types satisfy

$$f_1 > \theta > f_2. \tag{30}$$

In addition, we impose the parameter restriction

$$\min\left\{\frac{v_1}{\rho} - \frac{v_2}{\rho + m(f_1 - \theta)}, \frac{v_2 - v_1}{\rho}\right\} > \bar{c}, \tag{31}$$

indicating that the upper bound of the support of the cost distribution $H(c)$ is not too large.[12]

Proposition 7. *Let the contest be an all-pay auction with stochastic attack costs satisfying conditions* (30) *and* (31) *and let* $n = 2$. *Then for suitable choice of the attack-cost distribution* $H(c)$ *there exists an equilibrium with positive sorting and there also exists an equilibrium with negative sorting.*

The proof of Proposition 7 considers sequences of cost distributions $(H^k)_{k=1}^{\infty}$ converging pointwise to a cost distribution H^* assigning probability one to the highest possible cost realization $\bar{c}$ and showing that for sufficiently large k the associated all-pay auctions with stochastic attack cost feature both an equilibrium with positive sorting and an equilibrium with negative sorting. The key to the argument is that for the case in which the attack cost is deterministic and given by $\bar{c}$ there exists both an equilibrium with positive sorting and an equilibrium with negative sorting with both of these equilibria featuring sorting by rents.[13] We explain here how to construct these two equilibria for the case of deterministic attack costs equal to $\bar{c} > 0$, leaving the remainder of the proof to Appendix B.

Let conditions (30) and (31) hold. We first argue that (x^*, g^*) with

$$x_1^* = \frac{v_1}{\rho} \quad \text{and} \quad x_2^* = \frac{v_2 + m\,[v_1/\rho + \bar{c}]\,[\theta - f_2]}{\rho + m[\theta - f_2]}, \tag{32}$$

as well as

$$g_1^* = \theta - f_2 \quad \text{and} \quad g_2^* = f_2 \tag{33}$$

[12] As our model also requires $\bar{c} > 0$, it is worthwhile pointing out that for any given $\bar{c} > 0$ and flow payoffs satisfying $2v_1 > v_2$, there exists $m > 0$, $\rho > 0$, $\theta \in (0, 1)$ and $f_1 \in (\theta, 1)$ such that (31) holds.

[13] The latter point is essential and is what leads to conditions (30) and (31). To see the issues involved, consider an all-pay auction with prohibitively high deterministic attack costs ($\bar{c} > v_2/\rho$ will suffice). It it then clear that no challenger will ever choose to attack. Incumbents are thus completely insulated from any challenges to their position. This implies that incumbency rents are given by $x_i^* = v_i/\rho$ for all i. Because there is no turnover, so that the balance conditions (15) hold vacuously, for any $g \in \mathcal{G}$ the tuple (x^*, g) is then an equilibrium. In particular, there are equilibria with positive sorting and with negative sorting. All the equilibria with negative sorting do, however, fail sorting by rents, indicating that these equilibria can not be approximated by a sequence of equilibria in all-pay auctions with attack costs satisfying Assumptions 1 - 3. We view sorting by rents as a key (and natural) feature of our model and find the possibility to obtain negative sorting from a failure of sorting by rents not very insightful.

is an equilibrium for an all-pay auction with deterministic attack cost $\bar{c} > 0$ that features both positive sorting and sorting by rents. Positive sorting is immediate from (33) as $g_2^*/f_2 = 1$ and $g_1^*/f_1 < 1$ is then implied by $\theta < 1$. Sorting by rents is somewhat less obvious. Observe that (32) implies

$$x_2^* - x_1^* = \frac{v_2 - v_1 + m\bar{c}(\theta - f_2)}{\rho + m(\theta - f_2)}, \tag{34}$$

from which it follows that $x_2^* - x_1 > \bar{c}$ is implied by $(v_2 - v_1)\rho > \bar{c}$, which we have assumed in (31). Therefore, we have

$$x_2^* - x_1^* > \bar{c}, \tag{35}$$

so that $x_2^* - x_1^* > 0$ holds and we have sorting by rent. It remains to establish that (32) and (33) indeed describe an equilibrium. Observe first that with strictly positive attack costs, it is never optimal for a challenger of type 1 to mount an attack (because $x_1^* - x_2^* < 0$), so that the incumbency rent x_1^* must be given by the continuation value of an incumbent of type 1. Such an incumbent is, as we have just noticed, never attacked by a challenger of type 1. Because $g_2^* = f_2$ holds, there are no challengers of type 2. Hence, an incumbent of type 1 is also never attacked by a challenger of type 2, so that the equilibrium condition for x_1^* is that it coincides with the present value of the incumbency flow payoff v_1, which is the first equation in (32). Given $x_1^* - x_2^* < 0$ and $g_2^* = f_2$, incumbents of type 2 will also never be attacked. Nevertheless, their incumbency rent is not simply v_2/ρ. The reason is that if an incumbent of type 2 were to loose the incumbency position, she would find it worthwhile to attack (only) incumbents of type 1 as doing so results in the expected gain $x_2^* - x_1^* - \bar{c}$, which is strictly greater than zero by (35). This yields the value equation $\rho x_2^* = v_2 - m[x_2^* - x_1^* - \bar{c}]g_1^*$, which can be rearranged to get x_2^* as specified in (32). Finally, given that there is no turnover in equilibrium, the specification of the incumbent distribution in (33) is clearly consistent with the balance conditions.[14]

Next, let $(\hat{x}, \hat{g})$ be given by

$$\hat{x}_1 = \frac{v_1}{\rho} \quad \text{and} \quad \hat{x}_2 = \frac{v_2}{\rho + m(f_1 - \theta)}, \tag{36}$$

as well as

$$\hat{g}_1 = \theta \quad \text{and} \quad \hat{g}_2 = 0. \tag{37}$$

Then $(\hat{x}, \hat{g})$ is an equilibrium featuring negative sorting and sorting by rents. Negative sorting is immediate from (37). Sorting by rent then follows by observing that the first inequality in (31) implies $\hat{x}_1 > \hat{x}_2$. To verify that $(\hat{x}, \hat{g})$ is an equilibrium first observe that $\hat{g}_2 = 0$ means that there are no incumbents of type 2. Therefore, challenger's of type 1 have a continuation value of zero. On the other hand, incumbents of type 1 will never be attacked because $\hat{x}_2 < \hat{x}_1$ holds. Hence, the incumbency rent $\hat{x}_1$ is the present value of v_1, as specified in (36). Given $\hat{x}_2 < \hat{x}_1$ it is also clear that challengers of type 2 will never attack. The incumbency rent $\hat{x}_2$ is thus determined by the continuation

[14]Note, however, that in contrast to the example discussed in the previous footnote, this specification of the incumbent distribution is the only one consistent with the given incumbency rents as any other $g \in \mathcal{G}$ would imply that incumbents of type 1 suffer a strictly positive loss from meetings with challengers of type 2, invalidating the first equation in (32). An analogous remark applies for the equilibrium with negative sorting that we consider in the following paragraph.

payoff an individual of type 2 would obtain if she were to hold an incumbency position. From (36) and the first inequality in (31) we have

$$\hat{x}_1 - \hat{x}_2 > \bar{c}, \tag{38}$$

so that such an individual would be attacked by any challenger of type 1 whom she happens to encounter. From (37) there are $(f_1 - \theta)$ of such challengers, so that the value equation for the incumbency rent of type 2 individuals becomes $\rho\hat{x}_2 = v_2 - m(f_1 - \theta)\hat{x}_2$, which is the second equality in (36). Finally, given that there is no turnover in equilibrium it is immediate that $(\hat{g}_1, \hat{g}_2)$ as given in (37) solves the balance conditions.

5 Conclusion

We have developed a model of repeated contests over incumbency positions among a population of players holding heterogeneous but commonly known valuations: Incumbents have to recurrently defend their positions against challengers, unsuccessful challengers continue searching for incumbency positions to contest, and defeated incumbents may regain incumbency positions again in the future by mounting successful challenges. We have identified conditions on the structure of the contests which ensure positive sorting in equilibrium and have shown that these conditions hold in the two standard complete-information contests, namely the all-pay auction and the Tullock contests. We have also established that under fairly general conditions positive sorting and incomplete surplus dissipation arise in the limit of continual conflicts. For the all-pay auction with two types we have shown uniqueness of equilibrium and have discussed how the frequency at which incumbency is contested affects the extend of sorting in this equilibrium. Finally, we have provided a non-trivial example of a contest which gives rise to an equilibrium with negative sorting because, somewhat paradoxically, individuals with lower flow payoffs from holding the incumbency position have higher incumbency rents in equilibrium.

Numerous open issues remain. For example, we believe that it should be possible to establish uniqueness of equilibrium much more generally than for the all-pay auction with two types, but so far we have been unable to do so. Obtaining such a uniqueness result for the Tullock contest would be of special interest – not only because of the prominence of this contest but also because it would provide the starting point for going beyond the comparative statics we have investigated in Section 4 to study, for instance, how the decisiveness of a contest affects the extend of sorting. Concerning extensions of our modelling framework, the literature on search-theoretic models of the labor market suggests a number of interesting possibilities. In particular, we view endogenizing the intensity at which challengers search for possibilities to acquire incumbency positions as a promosing direction to pursue as we conjecture that doing so will have a profound impact on the extend of sorting arising in equilibrium.

A Proofs for Section 3

Proof of Proposition 1. Fix ϵ satisfying $0 < \epsilon < v_1/(\rho + m)$. Let $\mathcal{X} = \{x \in \mathbb{R}^n_+ : \epsilon \leq x_i \leq v_i/\rho, \forall i \in N\}$ and recall $\mathcal{G} = \{g \in \mathbb{R}^n_+ : g_i \leq f_i, \forall i \in N \text{ and } \sum_{i=1}^n g_i = \theta\}$. Define the vector-valued maps $V : \mathcal{X} \times \mathcal{G} \to \mathcal{X}$ and $G : \mathcal{X} \times \mathcal{G} \to \mathcal{G}$ by setting

$$V_i(x, g) = \max \left\{ \frac{v_i - m \sum_{j \in N} \tau(x_i, x_j)(f_j - g_j) - m \sum_{j \in N} \sigma(x_i, x_j) g_j}{\rho}, \epsilon \right\} \tag{39}$$

$$G_i(x, g) = g_i + (f_i - g_i) \sum_{j \in N} g_j \mu(x_i, x_j) - g_i \sum_{j \in N} (f_j - g_j) \mu(x_j, x_i). \tag{40}$$

Because $\sigma(x_i, x_j) \geq 0$ and $\tau(x_i, x_j) \geq 0$ holds for all $x \in \mathcal{X}$, it is immediate that V does indeed map into $\mathcal{X}$. To verify that G maps into $\mathcal{G}$ observe that the inequalities $0 \leq \mu(y, z) \leq 1$, which hold for all $(y, z) \in \mathbb{R}^2_{++}$ imply

$$G_i(x, g) \geq g_i - g_i(1 - \theta) = g_i \theta \geq 0$$

and

$$G_i(x, g) \leq g_i + (f_i - g_i)\theta = \theta f_i + (1 - \theta) g_i \leq f_i,$$

for all $(x, g) \in \mathcal{X} \times \mathcal{G}$. Further, adding (40) over all i, gives $\sum_{i=1}^n W_i(x, g) = \sum_{i=1}^n g_i = \theta$.

The sets $\mathcal{X}$ and $\mathcal{G}$ are both non-empty, compact and convex and, by continuity of σ, τ, and μ, the functions V and G are both continuous. Hence, Brouwer's fixed point theorem implies that the mapping $(V, G) : \mathcal{X} \times \mathcal{G} \to \mathcal{X} \times \mathcal{G}$ has a fixed point. Comparing (39) with (18) and (40) with (15) it is immediate that any such fixed point (x^*, g^*) is an equilibrium, provided that the inequalities

$$x_i^* = V_i(x^*, g^*) > \epsilon \tag{41}$$

holds for all $i \in N$. We now argue that given our choice of $\epsilon < v_1/(\rho + m)$ this is the case. For suppose (41) does not hold for some $i \in N$. For such i we then have $x_i^* = \epsilon$. Using the upper bounds (2) from Assumption 2 yields the inequality

$$v_i - m \sum_{j \in N} \tau(\epsilon, x_j)(f_j - g_j) - m \sum_{j \in N} \sigma(\epsilon, x_j) g_j \geq v_i - m\epsilon,$$

so that $\epsilon = V_i(x^*, g^*)$ implies $\epsilon \geq (v_i - m\epsilon)/\rho$ or, equivalently, $\epsilon \geq v_i/(m + \rho)$. Consequently, for $\epsilon < v_1/(\rho + m)$ every fixed point of (V, G) is an equilibrium. □

Proof of Lemma 1. Fix $x \in \mathbb{R}^n_{++}$. From Assumption 3 we have that $\mu(x_i, x_j) > 0$ holds for all $i, j \in N$ (as otherwise there exists $(x_I, x_C) \in \mathbb{R}^2_{++}$ such that $\mu(x_I, x_C) = 0$ holds, with Assumption 3 then implying that μ takes on strictly negative values on its domain, which is impossible.) Observing that $g_i = 0$ implies that the right side of the balance condition (15) for type i is equal to zero, whereas the left side is strictly positive for $g \in \mathcal{G}$ implies that if $(x, g) \in \mathcal{R}^n_{++} \times \mathcal{G}$ solves (15), then g is strictly positive.

Let $\mathcal{D} = \prod_{i \in N} [0, f_i]$ and let $D_{-k} = \prod_{i \neq k \in N} [0, f_i]$ for $k \in N$. Define the map $F : \mathcal{D} \to \mathbb{R}^n$ by letting its $i - th$ coordinate be given by the net outflow of type i from the incumbent position, that is,

$$F_i(g) = \sum_{j \in N} \left[g_i (f_j - g_j) \mu(x_i, x_j) - (f_i - g_i) g_j \mu(x_j, x_i) \right], \tag{42}$$

for all $i \in N$. By construction, $g \in \mathcal{D}$ then solves the balance conditions (15) if and only $F(g) = 0$ holds. A straightforward application of Brouwer's fixed point theorem along the lines given in the proof of Proposition 1 shows that for any $\theta \in (0, 1)$ such a solution satisfying $\sum_{j\in N} g_j = \theta$, so that $g \in \mathcal{G}$ holds, exists and that, similarly, for any given $\bar{g}_k \in [0, f_k]$ there exists a vector $g \in \mathcal{D}$ satisfying $g_k = \bar{g}_k$ and solving $F(g) = 0$. It remains to show that for every $\theta \in (0,1)$ there is exactly one solution to the equation $F(g) = 0$ satisfying $\sum_{j\in N} g_j = \theta$.

Towards this end, we begin by observing that the Jacobian matrix $J(g)$ of F is defined for all $g \in \mathcal{D}$ and is continuous in g. Let $J_{-k}(g)$ denote the Jacobian-matrix with the $k-th$ row and column deleted. Straightforward calculation shows that - because $\mu(x_i, x_j) > 0$ holds for all $i, j \in N$ the matrix $J_{-k}(g)$ has a positive strictly (column) dominant diagonal and negative non-diagonal entries. Hence, $J_{-k}(g)$ is a P-matrix and a Leontief matrix (Gale and Nikaido, 1965). As the domain $\mathcal{D}_{-k}$ is rectangular, it follows (Gale and Nikaido, 1965) that for any given $\bar{g}_k \in [0, f_k]$ there is exactly one solution $g \in \mathcal{D}$ to the equation $F(g) = 0$ satisfying $g_k = \bar{g}_k$. Using $H : [0, f_k] \to \mathcal{D}_{-k}$ to denote the function mapping $\bar{g}_k$ into the remaining coordinates of the vector g solving $F(g) = 0$, it is easy to see that $\sum_{j\neq k} H_j(0) = 0$ and $\sum_{j\neq k} H_j(f_k) = \sum_j f_j = 1 - f_k$ holds. Further, the implicit function theorem ensures that H is continuous and - because $J_{-k}(g)$ is a Leontief matrix, so that all of the elements of its inverse are positive (Gale and Nikaido, 1965) - increasing. It follows that the function $Z : [0, f_k] \to \mathbb{R}_+$ defined by $Z(\bar{g}_k) = \sum_j H_j(\bar{g}_k) + \bar{g}_k$ is continuous, strictly increasing, and satisfies $Z(0) = 0$ and $Z(1) = 1$. Consequently, for every $\theta \in (0,1)$ all solutions to the equation $F(g) = 0$ satisfying $\sum_j g_j = \theta$ have the property that g_k is given by the unique solution to the condition $Z(g_k) = \theta$. As this argument applies for all $k \in N$, the desired uniqueness result follows. □

Proof of Lemma 2. Let (x, g) be an equilibrium. As noted at the beginning of the proof of Lemma 1, Assumption 3 implies that $\mu(x_i, x_j) > 0$ holds for all $i, j \in N$. We also have $\sum_{j\in N} g_j = \theta > 0$ and $\sum_{j\in N}(f_j - g_j) = (1-\theta) > 0$. Hence, together with $g_j \geq 0$ and $f_j - g_j \geq 0$ for all $j \in N$, we obtain

$$\sum_{j\in N} g_j \mu(x_j, x_i) > 0 \text{ and } \sum_{j\in N}(f_j - g_j)\mu(x_i, x_j) > 0.$$

for all $i \in N$. Therefore, we may rearrange the flow balance conditions (15) as

$$\frac{g_i}{f_i - g_i} = \frac{\sum_{j\in N} g_j \mu(x_j, x_i)}{\sum_{j\in N}(f_j - g_j)\mu(x_i, x_j)} \tag{43}$$

for all $i \in N$. By Assumption 3, each summand in the numerator of the right hand expression in (43) is strictly increasing in x_i and each summand in the denominator is strictly decreasing in x_i. Hence, we obtain $x_i > x_j \Rightarrow g_i/(f_i - g_i) > g_j/(f_j - g_j) \Leftrightarrow g_i/f_i > g_j/f_j$, which is the desired conclusion. □

Proof of Lemma 3. Let (x, g) be an equilibrium. From the bounds (2) in Assumption 2 and the value equations in (18) we obtain

$$v_i \geq \rho x_i \geq v_i - m x_i \Leftrightarrow x_i \in \left[\frac{v_i}{\rho + m}, \frac{v_i}{\rho}\right].$$

Hence, $x_{i+1} > x_i$ is implied by

$$\frac{v_i}{\rho} < \frac{v_{i+1}}{\rho + m} \Leftrightarrow m < \rho\left[\frac{v_{i+1}}{v_i} - 1\right].$$

By definition of $\underline{m}$, the latter inequality holds for $i = 1, \dots n-1$ whenever $m < \underline{m}$, delivering the result. $\square$

Proof of Lemma 4. Deducting the value equations (18) for two adjacent types $i, i+1 \in N$ from each other, we obtain

$$m \sum_{j \in N} \left[\left[\tau(x_{i+1}, x_j) - \tau(x_i, x_j) \right] (f_j - g_j) + \left[\sigma(x_j, x_{i+1}) - \sigma(x_j, x_i) \right] g_j \right] + \rho \left[x_{i+1} - x_i \right]$$
$$= v_{i+1} - v_i.$$

The right side of this expression is strictly positive, whereas for $x_i \geq x_{i+1}$ the assumption that $\sigma(x_I, x_C)$ is increasing in x_C and $\tau(x_I, x_C)$ is increasing in x_I implies that the left side is negative. Hence, in every equilibrium $x_{i+1} > x_i$ must hold for all $i, i+1 \in N$. $\square$

Proof of Corollary 1. As noted before the statement of the corollary, it it obvious from (5) and (6) that for all the all-pay auction $\sigma(x_I, x_c)$ is increasing in x_C and $\tau(x_I, x_C)$ in increasing in x_I, so that the result follows from Proposition 2.

For the Tullock contest Nti (1999) calculates

$$\sigma_2(x_I, x_C) = \frac{1}{\left(x_I^r + x_C^r\right)^3} \left[\left(x_C^{2r} + (r+1) x_C^r x_I^r \right) \left(x_C^r + (1-r) x_I^r \right) + r^2 x_C^{2r} x_I^r \right],$$

which is clearly positive for $0 \leq r \leq 1$.[15] Further, $\sigma_2(x_I, x_C) \leq 1$ is equivalent to

$$x_C^{3r} + (2 + r^2) x_C^{2r} x_I^r + (r+1)(1-r) x_C^r x_I^{2r} \leq (x_I^r + x_C^r)^3,$$

which is also satisfied for $0 \leq r \leq 1$ and $(x_I, x_C) \in \mathbb{R}^2_{++}$. $\square$

Proof of Lemma 5. Let (x, g) be a limit equilibrium and let $(m^k, x^k, g^k)_{k=1}^{\infty}$ be the associated sequence with the properties indicated in Definition 5.

From the value equations (18) and the requirement that (x^k, g^k) is an equilibrium when the meeting rate is m^k we have

$$\rho x_i^k + m^k \left[\sum_j \tau(x_i^k, x_j^k)(f_j - g_j^k) + \sigma(x_j^k, x_i^k) g_j^k \right] = v_i \tag{44}$$

for all $i \in N$. We now consider the the two cases appearing in the statement of the lemma.
(i) Equation (44) implies

$$\rho x_i^k + m^k \left[\tau(x_i^k, x_i^k)(f_i - g_i^k) + \sigma(x_i^k, x_i^k) g_i^k \right] \leq v_i.$$

Using the inequality $\tau(y, y) \geq \sigma(y, y)$ implied by (3) in Assumption 2 this implies

$$\rho x_i^k + m^k \sigma(x_i^k, x_i^k) f_i \leq v_i. \tag{45}$$

[15]In fact, the expression is positive whenever the condition for existence of a pure strategy equilibrium in a Tullock contest is satisfied.

Suppose there exists some $i \in N$ such that the sequence $(x_i^k)_{k=1}^\infty$ does not converge to zero, so that it converges to some $x_i^* > 0$.

As $\sigma(x_i^*, x_i^*) > 0$ holds by assumption and $(m^k)_{k=1}^\infty$ converges to infinity, it then follows that the left side of (45) converges to infinity, yielding a contradiction. Therefore $\lim_{k\to\infty} x_i^k = 0$ holds for all $i \in N$.
(ii) Define

$$\zeta = \min_{g \in \mathcal{G}} Z(g),$$

where

$$Z(g) = \max_{i \in N} \min\{g_i, f_i - g_i\}.$$

The function $Z : \mathcal{G} \to \mathbb{R}_+$ is clearly continuous and, using the genericity condition (13), strictly positive on its domain. As $\mathcal{G}$ is compact, it follows that ζ is not only well-defined, but satisfies $\zeta > 0$.

Considering that all the summands multiplying m^k in (44) are positive and that $Z(g^k) \geq \zeta$ holds, we obtain the existence of j^k such that

$$\rho x_i^k + m^k \left[\tau(x_i^k, x_{j^k}^k) + \sigma(x_{j^k}^k, x_i^k)\right] \zeta \leq v_i \tag{46}$$

holds for all i. Using the assumption of role symmetry, the term in square brackets in (46) is identical to x_i^k, delivering the inequality

$$[\rho + m^k \zeta] x_i^k \leq v_i$$

for all i and k. As $\zeta > 0$ holds and the sequence $(m^k)_{k=1}^\infty$ converges to infinity, the desired conclusion $\lim_{k\to\infty} x_i^k = 0$ for all i is then immediate. □

Proof of Proposition 3. Let (x, g) be a limit equilibrium and let $(m^k, x^k, g^k)_{k=1}^\infty$ be the associated sequence with the properties indicated in Definition 5. As the conditions of Lemma 5 are satisfied, we have $x = 0$.

Using homogeneity of the contest, we may rewrite the balance equations as

$$(f_i - g_i^k) \sum_{j \in N} g_j^k \mu(x_j^k / x_i^k, 1) = g_i^k \sum_{j \in N} (f_j - g_j^k) \mu(1, x_j^k / x_i^k), \ \forall i \in N. \tag{47}$$

Similarly, the value equations can be rewritten as

$$x_i^k \left[\rho + m^k Z_i(x^k)\right] = v_i, \tag{48}$$

where

$$Z_i(x^k) = \sum_{j \in N} \left[\tau(1, \frac{x_j^k}{x_i^k})(f_j - g_j^k) + \sigma(\frac{x_j^k}{x_i^k}, 1) g_j^k\right], \tag{49}$$

Observe that (48) implies

$$\frac{x_j^k}{x_i^k} = \left(\frac{v_j}{v_i}\right) \left(\frac{\rho/m^k + Z_i(x^k)}{\rho/m^k + Z_j(x^k)}\right). \tag{50}$$

From the bounds (2) in Assumption 2 we have that $Z_i(x^k)$ as defined in (49) is bounded above by 1. Further (cf. the proof of Lemma 5), under condition (i) of Lemma 5 $Z_i(x^k)$ is bounded below by $\sigma(1,1) f_i > 0$ and under condition (ii) of Lemma 5 it is bounded below by $\zeta > 0$. No matter which of these two lower bounds is applicable, we thus obtain from (50) that there exists $\underline{\alpha} > 0$ and $\overline{\alpha} > \underline{\alpha}$ such that

$$\alpha_{ij}^k = \frac{x_j^k}{x_i^k} \in [\underline{\alpha}, \overline{\alpha}]$$

holds for all $i, j \in N$ and k. Consequently, the sequence $(m^k, x^k, g^k)_{k=1}^{\infty}$ has a subsequence for which all the ratios x_j^k / x_i^k converge to some finite limit $\alpha_{ij} > 0$. Taking limits along this subsequence and using (47), we then obtain that the limit equilibrium (x, g) satisfies

$$(f_i - g_i) \sum_{j \in N} g_j \mu(\alpha_{ij}, 1) = g_i \sum_{j \in N} (f_j - g_j) \mu(1, \alpha_{ij}), \ \forall i \in N.$$

Provided that α_{ij} is strictly increasing in j, the desired result then follows by applying the same argument as in the proof of Lemma 2.

As the conditions from Proposition 2 are satisfied, we have that $j > i$ implies $x_j^k > x_i^k$ for all k. Consequently, we have $\alpha_{ij} \geq 1$ for all $j > i$. As $\alpha_{ij} > 1$ for all $j > i$ implies that α_{ij} is strictly increasing in j (because for $\ell > j > i$ we have $\alpha_{i\ell} = \alpha_{ij}\alpha_{j\ell}$), it thus remains to exclude the possibility that $\alpha_{ij} = 1$ holds for some $i \neq j$. Suppose to the contrary that such a pair of types exists. As $\alpha_{ij} = 1$ implies $\alpha_{i\ell} = \alpha_{j\ell}$ for all $\ell \in N$, (49) then implies that along the relevant subsequence $Z_i(x^k)$ and $Z_j(x^k)$ converges to the same limit. Taking limits in (50) along the same subsequence, we then obtain $\alpha_{ij} = v_j / v_i$. As $v_j \neq v_i$ holds for $i \neq j$, this contradicts the hypothesis $\alpha_{ij} = 1$, finishing the proof. □

Proof of Proposition 4. Let (x, g) be a limit equilibrium and let $(m^k, x^k, g^k)_{k=1}^{\infty}$ be the associated sequence with the properties indicated in Definition 5. As the conditions of Lemma 5 are satisfied, we have $x = 0$. Let $(S^k)_{k=1}^{\infty}$ be the associated sequence of surpluses as defined by (20) and suppose that there exists a subsequence, which we may take to be the sequence $(m^k, x^k, g^k)_{k=1}^{\infty}$, itself such that $(S^k)_{k=1}^{\infty}$ converges to zero. We argue that this results in a contradiction.

Using the arguments from the proof of Proposition 3, we may suppose without loss of generality that the sequences $(\alpha_{ij}^k)_{k=1}^{\infty}$ defined by $\alpha_{ij}^k = x_j^k / x_i^k$ converge to finite, strictly positive limits α_{ij} that satisfy $\alpha_{ij} < 1$ for $i > j$.

Using homogeneity, we may rewrite the expression $\sum_{j \in N} \sigma(x_j^k, x_n^k) g_j^k$ as

$$x_n^k \sum_{j \in N} \sigma(\alpha_{nj}^k, 1) g_j^k$$

and then observe that

$$S^k \geq f_n \frac{m^k}{\rho} x_n^k \sum_{j \in N} \sigma(\alpha_{nj}^k, 1) g_j^k$$

holds. Consequently, to obtain the desired contradiction, it suffices to show that

$$\lim_{k \to \infty} m^k x_n^k \sum_{j \in N} \sigma(\alpha_{nj}^k, 1) g_j^k > 0. \tag{51}$$

To establish (51) we proceed as follows. Taking limits in equation (49) appearing in the proof of Proposition 3 for type n, we then have

$$Z_n = \lim_{k\to\infty} Z_n(x^k) = \sum_{j\in N} [\tau(1,\alpha_{nj})(f_j - g_j) + \sigma(\alpha_{nj},1)g_j] \tag{52}$$

As g satisfies (cf. the proof of Proposition 3)

$$(f_i - g_i)\sum_{j\in N} g_j\mu(\alpha_{ij},1) = g_i \sum_{j\in N}(f_j - g_j)\mu(1,\alpha_{ij}),\ \forall i \in N$$

and the winning probabilities appearing in this expression are all strictly positive (cf. the initial paragraph in the proof of Lemma 1), we have $g_j > 0$ holds for all $j \in N$. Further, because $\alpha_{nj} < 1$ holds for $j < n$ and α_{nj} thus increases in j, we have that $r = \max_{j<n}\alpha_{nj} < 1$ holds. By the assumption that $\sigma(y,z) > 0$ holds for all $y < z$, this implies

$$\sum_{j\in N}\sigma(\alpha_{nj},1)g_j \geq \sigma(r,1)\sum_{j\neq n} g_j > 0. \tag{53}$$

Comparing the left side of (53) with the left side of (51), it then suffices to show that the term $m^k x_n^k$ does not converge to zero to establish (51). Noting that, as established in the proof of 3, $Z_n(x^k)$ is bounded above by 1, this is, however, immediate from (48) which yields

$$x_n^k\left[\rho + m^k Z_n(x^k)\right] = v_n,$$

thus finishing the proof. □

B Proofs for Section 4

Proof of Proposition 5. Let (x_1, x_2) satisfying $x_2 > x_1 > 0$ solve (23) – (24) for given $g_1 > 0$. To simplify notation let $\alpha = x_2/x_1$.

It is readily verified that (23) – (24) imply

$$[\alpha - 1][\rho + m(1-\theta) + m(2g_1 - f_1)]v_1 = [v_2 - v_1][\rho + m(1-\theta)]. \tag{54}$$

Substituting $g_2 = \theta - g_1$ from (25) into (22) yields

$$(\theta - g_1)(f_1 - g_1) = [2\alpha - 1]g_1(f_2 - \theta + g_1),$$

which is equivalent to

$$f_1\theta - g_1 = 2[\alpha - 1][g_1(f_2 - \theta) + g_1^2]. \tag{55}$$

Using (54) to substitute for $\alpha - 1$ into (55) yields an equilibrium condition that only depends on g_1, namely

$$[f_1\theta - g_1][\rho + m(1-\theta) + m(2g_1 - f_1)]v_1 = 2[v_2 - v_1][\rho + m(1-\theta)][g_1(f_2 - \theta) + g_1^2]. \tag{56}$$

As existence of an equilibrium is assured by Proposition 1, it suffices to establish that (56) cannot have more than one solution in the interval $\mathcal{G}_1 = [\max\{\theta - f_2, 0\}, \min\{f_1, \theta\}]$. To do so we distinguish two cases: (i) $f_2 - \theta \geq 0$ and (ii) $f_2 - \theta < 0$.

We begin with case (i). Because $f_2 - \theta \geq 0$, we have $\mathcal{G}_1 = [0, \min\{f_1, \theta\}]$. At $g_1 = 0$

the right side of (56) is zero, and the left side is $f_1\theta[\rho + m[f_2 - \theta]] > 0$. Furthermore, because the left side of (56) is concave in g_1 and the right side is convex g_1, there can only be one $g_1 \geq 0$ solving (56).

We turn to case (ii). Because $f_2 - \theta < 0$, we have $\mathcal{G}_1 = [\theta - f_2, \min\{f_1, \theta\}]$. Observe that the right side of (56) has one root at $g_1 = 0$ and a second one at $g_1 = \theta - f_2 > 0$. The right side of (56) is strictly increasing at $g_1 = \theta - f_2$ and hence, because it is convex in g_1, it is strictly increasing at all $g_1 > \theta - f_2$. Next, we note that the left side of (56) has one root at $g_1 = [\theta - f_2 - \rho/m]/2 < \theta - f_2$ and one root at $g_1 = f_1\theta > \theta - f_2$. Because the left side of (56) is concave and strictly decreasing at the root $g_1 = f_1\theta$, it follows that there is a unique g_1 satisfying $\theta - f_2 < g_1 < f_1\theta < \min\{f_1, \theta\}$ that solves (56). □

Proof of Proposition 6. The result for case (ii) has already been established in the text. We consider case (i) here; the argument for case (iii) is analogous.

Let $\theta < \theta^*$. As in the proof of Proposition 5 we use (25) to rewrite the balance condition (22) as

$$(\theta - g_1)(f_1 - g_1) = [2\alpha - 1]g_1(f_2 - \theta + g_1). \tag{57}$$

for $\alpha \geq 1$. By Lemma 1, this equation has a unique solution $g_1(\alpha) \in \mathcal{G}_1 = [0, \min\{f_1, \theta\}]$.

Next, let

$$F(\alpha, m) = \frac{v_2}{v_1}\left(\frac{\rho + m(1-\theta)}{\rho + m(1-\theta) + m\,(1 - 1/\alpha)\,(2g_1(\alpha) - f_1)}\right) - \alpha, \tag{58}$$

As (23) and (24) imply

$$\frac{x_2}{x_1} = \frac{v_2}{v_1}\left(\frac{\rho + m(1-\theta)}{\rho + m(1-\theta) + m\,(1 - x_1/x_2)\,(2g_1 - f_1)}\right),$$

it is easily verified that if (x_1, x_2, g_1, g_2) is an equilibrium for meeting rate $m > 0$, then $\alpha = x_2/x_1$ solves

$$F(\alpha, m) = 0. \tag{59}$$

Vice versa, if α solves (59), then letting x_1 be the unique solution to (23), setting $x_2 = \alpha x_1$, $g_1 = g_1(\alpha)$ and $g_2 = \theta - g_1(\alpha)$ provides a solution to the equilibrium conditions. Hence, from the uniqueness result in Proposition 5, equation (59) has a unique solution $\alpha(m)$ satisfying $\alpha(m) > 1$. As $F(1, m) > 0$ holds, this solution must occur at a point where $F(\alpha, m)$ intersects 0 from above. Inspection of (58) reveals that for $\alpha > 1$ the partial derivative of F with respect to m satisfies

$$F_m(\alpha, m)\begin{cases} > 0 & \text{if } 2g_1(\alpha) < f_1 \\ = 0 & \text{if } 2g_1(\alpha) = f_1 \\ < 0 & \text{if } 2g_1(\alpha) > f_1 \end{cases}.$$

It follows that $\alpha(m)$ is strictly increasing in m if $2g_1(\alpha(m)) < f_1$ holds. Further, as observed in the text, $g_1(\alpha)$ is clearly strictly decreasing in α and $g_2(\alpha)$ is strictly increasing in α. Therefore, to finish our argument, it suffices to show that $2g_1(\alpha(m)) < f_1$ holds for all $m > 0$. In fact, as $g_1(\alpha(m))$ is strictly decreasing in m whenever $2g_1(\alpha(m)) < f_1$ holds, it suffices to show this for m sufficiently small.

Let $\alpha^* = \lim_{m\to 0}\alpha(m)$. Because $g(m) \in \mathcal{G}_1$ holds for all $m > 0$, it is easy to see from (23) and (24) that this limit exists and satisfies $\alpha^* = v_2/v_1$. Now, by definition of

$g_1(\alpha)$, we have that $g_1(\alpha^*)$ satisfies (57), so that we obtain

$$(\theta - g_1(\alpha^*)(f_1 - g_1(\alpha^*) = [2\alpha^* - 1]g_1(\alpha^*(f_2 - \theta + g_1(\alpha^*)),$$

which we may rewrite as

$$\frac{\theta - g_1(\alpha^*)}{1 - f_1 - \theta + g_1(\alpha^*)} = [2\alpha^* - 1]\frac{g_1(\alpha^*)}{f_1 - g_1(\alpha^*)}.$$

From this equality we see that

$$2g_1(\alpha^*) < f_1 \Leftrightarrow \frac{\theta - f_1/2}{1 - \theta - f_1/2} < [2\alpha^* - 1] \Leftrightarrow \theta < \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{\alpha^*}\right)(1 - f_1),$$

and, consequently,

$$2g_1(\alpha^*) < f_1 \Leftrightarrow \theta < \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{\alpha^*}\right)(1 - f_1) = \theta^*,$$

where the last equality follows from the definition of θ^* in (26) and $\alpha^* = v_2/v_1$. Therefore $\theta < \theta^*$ implies that $2g_1(\alpha(m)) < f_1$ holds for all sufficiently small $m > 0$. □

Proof of Proposition 7. We consider a sequence of all-pay auctions with attack costs parameterized by a sequence of attack-cost distributions $(H^k)_{k=1}^{\infty}$ with distributions H^k that are continuous on their support $[0, \bar{c}]$ and have a mass point at zero. All other relevant parameters are held fixed and are assumed to satisfy conditions (30) and (31). The sequence $(H^k)_{k=1}^{\infty}$ is assumed to converge pointwise to $H^*(x) = \mathbb{1}_{\{x \geq \bar{c}\}}$, corresponding to the case of deterministic attack costs discussed in the main body of the paper. For each of the two equilibria (x^*, g^*) and $(\hat{x}, \hat{g})$ for this deterministic case we show that there exists a sequence $(x^k, g^k)_{k=1}^{\infty}$, where (x^k, g^k) is an equilibrium for the all-pay auction with cost distribution H^k, converging to the limit (x^*, g^*), resp. to the limit $(\hat{x}, \hat{g})$. As the conditions for positive (resp. negative) sorting and for sorting by rents are given by strict inequalities this suffices to prove the proposition. Throughout the following we use E^k to denote expectations with respect to the distribution H^k.

We start with the positive sorting equilibrium discussed in the text, i.e. we want to argue that for our sequence $(H^k)_{k=1}^{\infty}$ converging to H^* we can construct a sequence $(x^k, g^k)_{k=1}^{\infty}$ of associated equilibria with limit (x^*, g^*) given by (32) and (33).

Consider (x^k, g^k) satisfying $x_2^k - x_1^k > \bar{c}$. It is then straightforward to verify that (x^k, g^k) is an equilibrium associated with the distribution H^k if

$$\rho x_1^k = v_1 - m\left[H^k(0)x_1^k(f_1 - g_1^k) + x_1^k(f_2 - g_2^k)\right] \tag{60}$$

$$\rho x_2^k = v_2 - m\left[H^k(0)x_1^k(f_1 - g_1^k) + H^k(0)x_2^k(f_2 - g_2^k) + [x_2^k - x_1^k - E^k[c]]g_1^k\right] \tag{61}$$

and

$$(f_1 - g_1^k)g_2^k H^k(0)\frac{1}{2}\frac{x_1^k}{x_2^k} = g_1^k(f_2 - g_2^k)\left[1 - \frac{1}{2}\frac{x_1^k}{x_2^k}\right] \tag{62}$$

hold.

Now consider replacing (62) by

$$(f_1 - g_1^k)g_2^k H^k(0)\frac{1}{2}\max\{\frac{x_1^k}{x_2^k}, 1\} = g_1^k(f_2 - g_2^k)\min\{\left[1 - \frac{1}{2}\frac{x_1^k}{x_2^k}\right], 1/2\}. \tag{63}$$

Then the existence argument from the proof of Proposition 1 is applicable to guarantee that equations (60), (61), and (63) have a solution $(x^k, g^k) \in \mathbb{R}^2_{++} \times \mathcal{G}$ for any k. We now argue that for sufficiently large k any such solution satisfies $x_2^k - x_1^k > \bar{c}$ and therefore solves (60) - (62) and thus is an equilibrium. Further, the sequence (x^k, g^k) converges to (x^*, g^*) as given by (32) and (33). To see this, observe that for $k \to \infty$ we have $H^k(0) \to 0$, so that the left side of (63) converges to zero. By the condition $\theta > f_2$ from (30), we have that g_1^k cannot converge to zero. Therefore (63) implies $g_2^k \to g_2^* = f_2$ and, as a consequence, $g_1^k \to g_1^* = \theta - f_2$. Observing that $E^k[c] \to \bar{c}$ holds, we thus obtain from (60) that x_1^k converges to x_1^* and from (61) that x_2^k converges to x_2^*. The strict inequality in (35) then establishes the desired result.

The argument for the existence of a sequence of equilibria $(x^k, g^k)_{k=1}^{\infty}$ of equilibria with limit $(\hat{x}, \hat{g})$ given by (36) and (37) is analogous: Consider (x^k, g^k) satisfying $x_1^k - x_2^k > \bar{c}$. It is then straightforward to verify that (x^k, g^k) is an equilibrium associated with the distribution H^k if

$$\rho x_1^k = v_1 - m\left[H^k(0)x_1^k(f_1 - g_1^k) + H^k(0)x_2^k(f_2 - g_2^k) + [x_1^k - x_2^k - E^k[c]g_2^k\right] \tag{64}$$

$$\rho x_2^k = v_2 - m\left[x_2^k(f_1 - g_1^k) + H^k(0)x_2^k(f_2 - g_2^k)\right], \tag{65}$$

and

$$(f_1 - g_1^k)g_2^k\left[1 - \frac{1}{2}\frac{x_2^k}{x_1^k}\right] = g_1^k(f_2 - g_2^k)H^k(0)\frac{1}{2}\frac{x_2^k}{x_1^k} \tag{66}$$

hold. Replacing (66) by

$$(f_1 - g_1^k)g_2^k\min\{\left[1 - \frac{1}{2}\frac{x_2^k}{x_1^k}\right], \frac{1}{2}\} = g_1^k(f_2 - g_2^k)H^k(0)\frac{1}{2}\max\{\frac{x_2^k}{x_1^k}, 1\} \tag{67}$$

we obtain the existence of a solution (x^g, g^k) to equations (64), (65), and (67). As the right side of (67) converges to zero for $k \to \infty$, the condition $f_1 > \theta$ from (30) implies that g_2^k converges to $\hat{g}_2 = 0$ and, therefore, g_1^k converges to $\hat{g}_1 = \theta$. Using $H_k(0) \to 0$ and $E^k[c] \to \bar{c}$, we then obtain $x_1^k \to x_1^*$ and $x_2^k \to x_2^*$. The result then follows from (38). □

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